Bernstein-type Results for Special Lagrangian Graphs

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Abstract

The celebrated theorem of Bernstein shows that the only entire minimal graphs in $\mathbb{R}^3$ must be planes. There are many efforts to generalize analogous Bernstein-type theorems to higher dimension, i.e. to anwser *when an entire minimal graph in $\mathbb{R}^n$ will be a plane.* For example, Simons [29] has proved that an entire minimal graph must be a plane for dimension lower than 7, whereas Bombieri, de Giorgi and Giusti [3] shortly after give a counter-example in dimension 8 and higher. Earlier before, Moser [26] had proved a Bernstein-type result in arbitrary dimension provided that the *slope* of the graph is uniformly bounded.

This thesis is an exposition of Bernstein-type results for special Lagrangian graphs. Such a graph $M \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$ is the graph of the gradient ∇F of a scalar smooth function $F : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^n$ an open domain. This graph is proved to be a Lagrangian submanifold of $\mathbb{C}^n$. This graph M is minimal or speical if and only if there exists a constant θ such that $\operatorname{Im} \det_{\mathbb{C}} e^{i\theta}(I + \operatorname{Hess} F) = 0$, in the sense of calibrated geometry by Harvey and Lawson (1982). Under certain conditions, the entire solution F satisfying $\operatorname{Im} \det_{\mathbb{C}} e^{i\theta}(I + \operatorname{Hess} F) = 0$, for some θ, is a quadratic polynomial, whence the minimal graph M is flat.

摘要

經典的 Bernstein 定理指出了只有平面才是在 $\mathbb{R}^3$ 裡唯一的整極小圖像。數學界不停努力推廣更高維數的 Bernstein 定理，那就是去尋找一個在 $\mathbb{R}^n$ 裡整極小圖像會是平面空間的條件。譬如， Simons [29] 就證明了在維數低於 7 的情況下，整極小圖像必須是平面空間；可是 Bombieri、de Giorgi 和 Giusti [3] 在不久之後就給出了在 8 維或更高維的一個反例。早在之前，Moser [26] 已經證明了在圖像斜率一致有界的條件下一個任意維的 Bernstein 結果。

本論文研究對於特殊拉格朗日圖像的 Bernstein 結果。這種圖像 $M \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$ 是由一個標量光滑函數 $F : \Omega \to \mathbb{R}$ 的梯度 ∇F 所構造，而當中 $\Omega \subseteq \mathbb{R}^n$ 是一個定義域。這樣的圖像已知是在 $\mathbb{C}^n$ 裡的一個拉格朗日子流形。依據 Harvey 和 Lawson (1982) 提出的標定幾何 (calibrated geometry)，我們稱圖像 M 為極小或特殊，當且僅當這存在一個常數 θ 使得 $\operatorname{Im} \det_{\mathbb{C}} e^{i\theta}(I + \operatorname{Hess} F) = 0$。在某些條件，整解 F 是一個二次多項式，而極小圖像 M 是一個平面空間。

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Contents

Chapter 1

Introduction

The celebrated Bernstein theorem shows that the only minimal graph in $\mathbb{R}^3$ must be a plane. The Bernstein problem – asking when a minimal graph in $\mathbb{R}^n$ will be a plane – is then raised. This thesis studies a generalization of this theorem in the case for special Lagrangian graph in $\mathbb{C}^n$ under bounded slope condition.

In §2.3, we will discuss that a graph M in $\mathbb{C}^n$ of a smooth map $f : \Omega \to \mathbb{R}^n$ is Lagrangian if and only if the matrix $\left(\frac{\partial f^i}{\partial x_j}\right)$ is symmetric. Furthermore if Ω is simply connected, $f = \nabla F$ for some smooth function $F : \Omega \to \mathbb{R}$. In terms of the calibrated geometry; cf. [15], the graph M of ∇F is called *special Lagrangian* in $\mathbb{C}^n$ if and only if there exists a constant θ such that

$$\operatorname{Im} \det_{\mathbb{C}} e^{i\theta}(I + i \operatorname{Hess} F) = 0. \tag{2.8}$$

Our main Bernstein-type result is now formulated as follows.

Theorem 6.3. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined on the whole $\mathbb{R}^n$. Assume that the graph of ∇F is a special Lagrangian submanifold M in $\mathbb{C}^n := \mathbb{R}^n \oplus \mathbb{R}^n$; cf.* (2.8). *Furthermore if there is a constant $\beta < \infty$ such that*

$$\Delta_F^2 := \det(I + (\operatorname{Hess} F)^2) \leq \beta^2, \tag{6.1}$$

then F is a quadratic polynomial and M is an affine n-plane.

Let us now observe that the special Lagrangian equation (2.8) is *similar* to the Monge-Ampère equation $\det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) = 1$, in which convex solution is concerned. Pogorelov [27] extended Calabi's result [4] to all n where any such convex solution F defined entirely on $\mathbb{R}^n$ has to be a quadratic polynomial.

In this direction, we add the convexity of F into our main result to get a simpler problem; cf. Theorem 6.1. With minimality of M in hand, we know, by Corollary 5.4 due to Ruh and Vilms, that the Gauss map γ is harmonic. Our first strategy is to apply the Liouville-type theorem, cf. Theorem 5.4, so as to show that $\gamma := \left(\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}\right) \in G(n,n)$, in terms of local coordinate (4.7), is constant, for then ∇F is linear and M is an affine n-plane in $\mathbb{R}^{n+m}$.

However, we need to consider that each tangent plane on M is Lagrangian subspace, where in §4.5 we will learn that the Lagrangian Grassmannian $\mathrm{Lag}(n)$ is a totally geodesic submanifold in $G(n,n)$. Let $\iota : \mathrm{Lag}(n) \hookrightarrow G(n,n)$ be the (Lagrangian) immersion, and without ambiguity, let $\gamma : M \to \mathrm{Lag}(n)$ denote the (Lagrangian) Gauss map, then the classical Gauss map $\iota \circ \gamma : M \to G(n,n)$ is harmonic, by composition formula and Corollary 5.2, γ is also harmonic.

In the *Proof of Theorem 6.1*, we will construct a suitable geodesically convex neighbourhood, cf. §4.4, around a fixed n-plane $P_0 \in \mathrm{Lag}(n)$ so that Theorem 5.4 can still apply due to the fact that F is convex together with the conditions for sectional curvatures of $\mathrm{Lag}(n)$; cf. §4.5. Hence we just improve our first strategy a little bit by showing that the (Lagrangian) Gauss map γ is constant so that M is an affine n-plane.

The relevant theory of Grassmann geometry will be reviewed in Chapter 4. There we will learn the Wong's formula (4.11) for geodesic segments in $G(n,m)$ and the Leichtweiss' formula (4.16) for curvature tensor at a given point in $G(n,m)$. Therefore we can then develop a technique to compute the sectional curvatures of $G(n,m)$ by regarding that $G(n,m)$ is a locally symmetric space. Thus an upper bound for the sectional curvatures can be easily found, a geodesically convex neighbourhood, cf. §4.4, which fits the Theorem 5.4, can then be constructed.

Later in Chapter 5, we will review the basic backgrounds for harmonic maps between Riemannian manifolds. Then we study the Gauss map of an n-manifold M embedded in $\mathbb{R}^{n+m}$ and its relation with the second fundamental form of M in $\mathbb{R}^{n+m}$. Furthermore we introduce the notion of *simple* Riemannian manifold and the Liouville-type theorem; cf. Theorem 5.4, which serves as our main weapon.

In Chapter 6, we will state Theorems 6.1 and 6.3 mentioned above. Moreover we will also study a spherical Bernstein-type result – Theorem 6.2, which helps proving Theorem 6.3, where it utilizes our findings in §3.3 to impose condition to the sectional curvatures of the manifold $C\tilde{M}$. We will then consider the tangent cone $C\tilde{M}$ of M at ∞, where $C\tilde{M}$ is a special Lagrangian cone of which the link $\tilde{M}$ is a compact minimal Legendrian submanifold in S^{2n-1}. In addition, if $C\tilde{M}$ is an affine plane, so is M by Allard's regularity estimate [1].

Chapter 2

Symplectic Geometry and Special Lagrangian Graphs in $\mathbb{C}^n$

In this chapter, we would investigate the symplectic geometry of $\mathbb{C}^n$ and study the special Lagrangian graphs in $\mathbb{C}^n$. Furthermore, we will briefly discuss the calibrated geometry introduced by Harvey and Lawson [15] in 1982. Some results here are also adapted from da Silva [8].

2.1 Symplectic and Lagrangian Geometry of $\mathbb{C}^n$

Let $\mathbb{C}^n$ denote the complex Euclidean n-space, with coordinates $z = (z_1, \ldots, z_n)$, where $z = x + iy$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Let $\mathbb{R}^n$ denote the subset of $\mathbb{C}^n$ where $y = 0$ with the standard orientation. Let $\omega := \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j$ denote the standard Kähler form on $\mathbb{C}^n$. Moreover we learn from symplectic geometry,

Definition 2.1. *A* ***symplectic manifold*** (X^{2n}, ω) *is a* $2n$*-manifolds with a* ***symplectic form*** ω*. A de Rham* 2*-form* ω *is called* ***symplectic*** *if* ω *is closed and* $\omega_x : T_xX \times T_xX \to \mathbb{R}$ *is symplectic* $\forall\, x \in X$.

By the way, one can put $\alpha = \sum_{j=1}^{n} y_j \wedge dx_j$, which is called *Liouville* 1*-form*; hence $d\alpha = -\omega$, implying that ω is closed. Thus $\mathbb{C}^n$ is a symplectic manifold with symplectic form ω.

Definition 2.2. *An oriented real* n*-plane* ζ *in* $\mathbb{C}^n$ *is called* ***totally real*** *if it contains no complex lines, i.e.* $u \in \zeta \;\Rightarrow\; Ju \notin \zeta$.

Definition 2.3. *An oriented real* n*-plane* ζ *in* $\mathbb{C}^n$ *is called* ***Lagrangian*** *if* $Ju \perp \zeta$ *for all* $u \in \zeta$ *is valid.*

Let $(\cdot\,,\cdot) := \sum_{j=1}^{n} dz_j \otimes d\bar{z}_j$ denote the standard Hermitian form on $\mathbb{C}^n$, let $\langle\cdot\,,\cdot\rangle := \sum_{j=1}^{n} dx_j^2 + dy_j^2$ denote the standard inner product on $\mathbb{C}^n$. They are related with ω by the formula

$$(u, v) = \langle\, u, v\,\rangle - i\omega(u, v),$$

for all vectors $u, v \in \mathbb{C}^n$.

Therefore $\langle\, Ju, v\,\rangle = \mathrm{Re}\,(Ju, v) = \mathrm{Re}\, i(u, v) = -\mathrm{Im}\,(u, v) = \omega(u, v)$. As a corollary, we may rephrase the definition of *Lagrangian* by

$$\omega|_\zeta \equiv 0. \tag{2.1}$$

Remark. In symplectic linear algebra, *Lagrangian subspace* is an ***isotropic subspace*** (i.e. subspace where ω vanishes) with dimension n. It is of maximal dimension due to dimension theorem.

Consider the Grassmannian $G(n, n)$ of oriented real n-planes in $\mathbb{R}^{2n} \cong \mathbb{C}^n$, and let $\mathrm{Lag}(n)$ denote the subset consisting of the Lagrangian planes, called ***Lagrangian Grassmannian***. One can easily check that the unitary group $\mathrm{U}(n)$ acts on $\mathrm{Lag}(n)$. Moreover, this action is *transitive*. Suppose that $\zeta := \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ and $\zeta' := \varepsilon'_1 \wedge \cdots \wedge \varepsilon'_n$ are Lagrangian with ε's denoting their orthonormal bases. Then $\{\, \varepsilon_1, \ldots, \varepsilon_n,\ J\varepsilon_1, \ldots, J\varepsilon_n\,\}$ and $\{\, \varepsilon'_1, \ldots, \varepsilon'_n,\ J\varepsilon'_1, \ldots, J\varepsilon'_n\,\}$ are both orthonormal bases for $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Consequently the linear map A such that $A\varepsilon_j = \varepsilon'_j$ is unitary and $A\zeta = \zeta'$. The isotropy subgroup of $\mathrm{U}(n)$ at the point $\zeta_0 := \mathbb{R}^n$ is $\mathrm{SO}(n)$ acting diagonally on $\mathbb{R}^n \oplus \mathbb{R}^n$. Thus

$$\mathrm{Lag}(n) \cong \mathrm{U}(n)/\mathrm{SO}(n). \tag{2.2}$$

We will study in detail the Riemannian geometry of $G(n,n)$ in Chapter 3.

2.2 Calibrated and Special Lagrangian Geometries in $\mathbb{C}^n$

Let X be a Riemannian manifold, and let $\varphi \in \Gamma(\wedge^p T^*X)$ be a closed exterior p-form on X with the property that

$$\varphi|_\xi \leq \mathrm{vol}_\xi \tag{2.3}$$

for all oriented tangent p-planes ξ on X. Then any compact oriented p-dimensional submanifold M of X with the property that

$$\varphi|_M = \mathrm{vol}_M \tag{2.4}$$

is homologically volume minimizing in X, i.e. $\mathrm{vol}(M) \leq \mathrm{vol}(M')$ for any M' such that $\partial M = \partial M'$ and $[M - M'] = 0$ in $H_p(X;\mathbb{R})$. To see this, one can note that

$$\mathrm{vol}(M) = \int_M \varphi = \int_{M'} \varphi \leq \mathrm{vol}(M'),$$

where the first equality follows from (2.4), the final inequality follows from (2.3), and the middle equality results from the homology condition and $d\varphi = 0$.

By (2.4), we associate to an exterior p-form φ a family of oriented p-submanifolds in X which we call ***φ-submanifolds***. If φ is closed and is *normalized* to satisfy (2.3), then the argument above proves that each φ-submanifold is homologically volume minimizing in X. This means that any compact φ-submanifold is a minimal submanifold of X. Harvey and Lawson [15] show also the noncompact case.

A closed exterior p-form φ satisfying (2.3) is called a ***calibration*** and the Riemannian manifold X together with this form is called a ***calibrated manifold***.

Definition 2.4. *An n-dimensional oriented submanifold M of $\mathbb{C}^n$ is a* ***special Lagrangian submanifold*** *of $\mathbb{C}^n$ if the tangent plane to M, at each point, is special Lagrangian.*

Definition 2.5. *An oriented n-plane ζ in $\mathbb{C}^n$ is called* ***special Lagrangian*** *if*

(1) *ζ is Lagrangian*

(2) *$\zeta = A\zeta_0$, where $A \in \mathrm{SU}(n)$ and $\zeta_0 := \mathbb{R}^n$ under standard orientation.*

Now for notational convenience we set

$$dz = dz_1 \wedge \cdots \wedge dz_n, \quad \alpha = \operatorname{Re} dz, \quad \beta = \operatorname{Im} dz$$

so that $dz = \alpha + i\,\beta$ with α and β real.

Now we have the following results from Harvey and Lawson [15].

Theorem 2.1. *$\alpha(\zeta) \leq |\zeta|$ for all $\zeta \in G(n,n)$ with equality if and only if ζ is special Lagrangian.*

Corollary 2.1. *Suppose $\zeta \in G(n,n)$. Then either ζ or $-\zeta$ is special Lagrangian if and only if*

(1) *ζ is Lagrangian, and*

(2) $\beta(\zeta) = 0$.

Moreover, if A is any complex linear map sending $\zeta_0 = e_1 \wedge \cdots \wedge e_n$ into $\lambda\zeta$ with $\lambda \in \mathbb{R}$, then $\lambda\beta(\zeta) = \mathrm{Im}\ \det_{\mathbb{C}} A$.

Corollary 2.1 implies that $\alpha := \mathrm{Re}\, dz = dz$ is a calibration on $\mathbb{C}^n$, which is called the ***special Lagrangian calibration***. Therefore, special Lagrangian submanifolds are α-submanifolds and thus are area minimizing.

2.3 Special Lagrangian Differential Equation

Suppose that M is a special Lagrangian submanifold of $\mathbb{C}^n$. Locally M can be described explicitly as the graph of a function over a tangent plane, by implicit function theorem. Since all special Lagrangian planes are equivalent, under $\mathrm{SU}(n)$, to the axis plane $\zeta_0 := \mathbb{R}^n$, we may consider M to be given as the graph, in $\mathbb{R}^n \oplus i\,\mathbb{R}^n = \mathbb{C}^n$, of a graphing function $y = f(x)$ where $z = x + iy$.

Lemma 2.1. *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^n$ is a C^1 mapping. Let M denote the graph of f in $\mathbb{C}^n = \mathbb{R}^n \oplus i\,\mathbb{R}^n$. Then the graph M is Lagrangian if and only if the Jacobian matrix $(\,\partial f^i/\partial x_j\,)$ is symmetric. In particular, if Ω is simply connected, M is Lagrangian if and only if $f = \nabla F$, is the gradient field of some potential function $F \in C^2(\Omega)$.*

Proof. We replace f by its Jacobian f_* at some fixed point. Then $f_* : \mathbb{R}^n \to \mathbb{R}^n$ is linear and its graph is of the form $TM = \{\, x + if_*(x) : x \in \mathbb{R}^n \,\}$. By definition, TM is Lagrangian if and only if $Jv \perp TM$ for all $v \in TM$. Suppose $v = x + i\,f_*(x)$. Then $Jv = -f_*(x) + ix$. Hence TM is Lagrangian if and only if $-f_*(x) + ix \perp x' + i\,f_*(x')$ for all $x, x' \in \mathbb{R}^n$, i.e. $-\langle\, f_*(x), x' \,\rangle + \langle\, x, f_*(x') \,\rangle = 0$ for all $x, x' \in \mathbb{R}^n$. Consequently M is Lagrangian if and only if the Jacobian matrix of f is symmetric at each point of Ω, or equivalently the form $f^j\,dx_j$ over $\mathbb{R}^n$ is closed. Since Ω is simply connected, this form is exact, i.e. $\exists$ a potential function $F : \Omega \to \mathbb{R}^n$ with $\nabla F = f$. $\square$

To study the special Lagrangian graph in $\mathbb{C}^n$, let

$$\operatorname{Hess} F := \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)$$

denote the Hessian matrix of $F : \Omega \to \mathbb{R}$ and let $\sigma_j(\operatorname{Hess} F)$ denote the j^{th} elementary symmetric polynomial of its eigenvalues.

Theorem 2.2. *Suppose $F \in C^2(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open. Let $f := \nabla F$ denote the gradient field, and let M denote the graph of f in $\mathbb{C}^n = \mathbb{R}^n \oplus i\,\mathbb{R}^n$. Then M is special Lagrangian if and only if*

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1}(\operatorname{Hess} F) = 0, \tag{2.5}$$

or equivalently,

$$\operatorname{Im} \det_{\mathbb{C}}(I + i \operatorname{Hess} F) = 0, \tag{2.6}$$

with the correct orientation.

Proof. Consider the tangent space ζ to M at a fixed point, by Lemma 2.1 we know $f_* : \mathbb{R}^n \to \mathbb{R}^n$ is linear and symmetric. Indeed, f_* is simply the linear map of which the matrix representation is $\operatorname{Hess} F$. The graph of f_* is the image of $\zeta_0 := e_1 \wedge \cdots \wedge e_n = \mathbb{R}^n$ under the complex linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ defined by $A := I + if_*$. It now follows immediately from Corollary 2.1 that M is special Lagrangian if and only if $\operatorname{Im} \det_{\mathbb{C}}(I + i \operatorname{Hess} F)$, i.e. (2.6) holds.

It remains only to show the equivalence of (2.5) and (2.6). Since the action of $\mathrm{SO}(n)$ on $\mathbb{C}^n$, given by $Q(x+iy) = Qx + iQy$, preserves the set of special Lagrangian n-planes, we may replace f_* by any linear map of the form $Q \circ f \circ Q^{-1}$ for $Q \in \mathrm{SO}(n)$, hence

$$\operatorname{Im} \det_{\mathbb{C}}(I + i\, f_*) = \operatorname{Im} \prod_{j=1}^{n} (1 + i\lambda_j) = \sum_k (-1)^k \sigma_{2k+1}(f_*).$$

Since the first and last terms are $\mathrm{SO}(n)$-invariant, this proves the equivalence. $\square$

Remark. As one may ask at the beginning, which orientation (ζ or $-\zeta$) is special Lagrangian? From the argument above, one can find also

$$\lambda\, \alpha(\zeta) := \operatorname{Re} \det_{\mathbb{C}}(I + i \operatorname{Hess} F) = \sum_{k=0}^{[n/2]} (-1)^k \sigma_{2k}(\operatorname{Hess} F).$$

($\sigma_0 := 1$ by convention.) The correct orientation for the graph of f in Theorem 2.2 can be determined by finding the sign of $\alpha(\zeta)$ from the equation above, i.e. $(\operatorname{sign} \alpha(\zeta)) \cdot \zeta$ is special Lagrangian.

The special Lagrangian Grassmannian is the fibre above $1 \in S^1$ of the Maslov map given by the complex determinant.

$$\mathrm{Lag}(n) \cong \mathrm{U}(n)/\mathrm{SO}(n) \xrightarrow{\det_{\mathbb{C}}} S^1 \tag{2.7}$$

This fibre is denoted by $\mathrm{SLag}(n)$.

Each fibre of the map (2.7) is the Grassmannian associated to a calibration on $\mathbb{R}^{2n}$. In fact, this family of Grassmannians belongs to the family of forms:

$$\alpha_\theta := \mathrm{Re}\, e^{i\theta} dz_1 \wedge \cdots \wedge dz_n$$

for $0 \leq \theta \leq 2\pi$. Thus we have an S^1-family of special Lagrangian geometries compatible with the given complex structure. Hence in general we conclude

Theorem 2.3. *Suppose $F \in C^2(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open. Let $f := \nabla F$ denote the gradient field, and let M denote the graph of f in $\mathbb{C}^n = \mathbb{R}^n \oplus i\,\mathbb{R}^n$. Then M is special Lagrangian if and only if $\exists$ constant θ such that*

$$\mathrm{Im}\,\det_{\mathbb{C}} e^{i\theta}(I + i\,\mathrm{Hess}\, F) = 0. \tag{2.8}$$

Chapter 3

Contact Geometry in S^{2n-1}

Contact geometry can be treated as the odd-dimensional analogue of symplectic geometry. In this chapter, we would study how the minimal Legendrian submanifold in $S^{2n-1} \subseteq \mathbb{C}^n$ is related to the special Lagrangian submanifold in $\mathbb{C}^n$. Results here are adapted from da Silva [8], Jost and Xin [21].

3.1 Contact and Legendrian Geometries in S^{2n-1}

Let $\mathbb{C}^n$ denote the complex Euclidean n-space, with coordinates $z = (z_1, \ldots, z_n)$, where $z = x + iy$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Let S^{2n-1} denote the subset of $\mathbb{C}^n$ where $|\,z\,| = 1$. Let $\omega := \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j$ denote the standard Kähler form and J denote the complex structure on $\mathbb{C}^n$.

Moreover, we have learnt from the contact geometry that

Definition 3.1. *A* ***contact element*** *on manifold X is a pair (x, H_x) where $x \in X$ is called the* ***contact point****, together with a tangent hyperplane $H_x \subseteq T_xX$ at x (not necessarily oriented), i.e. a codimension one subspace of T_xX.*

The tangent hyperplane $H_x \subseteq T_xX$ determines a covector $\eta_x \in T_x^*X - \{0\}$, up to multiplication by a nonzero scalar. Indeed, (x, H_x) is a contact element if and only if $\exists\ \eta_x \in T_x^*X$ such that $H_x = \ker \eta_x$ and $\eta_x \neq 0$. Suppose H is a smooth field of tangent hyperplanes of contact elements on X, i.e. $H : x \mapsto H_x \subseteq T_xX$. Locally, $H = \ker \eta$ for some 1-form η, called ***locally defining 1-form*** for H.

Definition 3.2. *A* ***contact structure*** *on X is a smooth field of tangent hyperplanes $H \in \Gamma(TX)$, such that for any locally defining 1-form η, we have $d\eta|_H$ nondegenerate. The pair (X, H) is then called a* ***contact manifold*** *and η is called a* ***contact form****.*

The distribution $H := \ker \eta$ is symplectic and has codimension 1, thus $(d\eta)^n|_H \neq 0$ is a volume form on H, which implies $\eta \wedge (d\eta)^n$ is a volume form on X.

Theorem 3.1. *Let H be a field of tangent hyperplanes on X and η be a locally defining* 1*-form for H. Then H is a contact structure if and only if $\eta \wedge (d\eta)^n \neq 0$.*

On the sphere, let $\vec{Z} := \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}$ be the position vector field of the sphere and η be the dual form of $J\vec{Z} := \sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}$ in S^{2n-1}, i.e.

$$\eta = \sum_{j=1}^{n} x_j dy_j - y_j dx_j.$$

Therefore, it is easily seen that

$$d\eta = 2\sum_{j=1}^{n} dx_j \wedge dy_j = 2\omega. \tag{3.1}$$

Indeed, we note that $d\left(\eta \wedge (d\eta)^{n-1}\right) = (d\eta)^n = 2^n \omega^n \neq 0$ on $\mathbb{C}^n$. This implies that $\eta \wedge (d\eta)^{n-1} \neq 0$ on S^{2n-1} and η is a contact form on S^{2n-1}, i.e. S^{2n-1} is a contact manifold.

Theorem 3.2. *Then $\exists$ a unique vector field $\vec{R}$ on S^{2n-1} such that $\iota_{\vec{R}}\, d\eta = 0$ and $\iota_{\vec{R}}\, \eta = 1$.*

This vector field $\vec{R}$ is called the ***Reeb vector field*** determined by η, which is $\sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}$. In contact geometry, we have

Definition 3.3. *Let (X, H) be a contact manifold with η its local contact form. A submanifold M of X is called an* ***integral submanifold*** *if $\eta(v) \neq 0$ for all $v \in TM$, or equivalently $T_xM \subseteq H_x$ for all $x \in M$.*

Definition 3.4. *An integral submanifold M of maximum dimension is called a* ***Legendrian submanifold****.*

If M is an integral submanifold of contact manifold (S^{2n-1}, H), it is obvious that for any $V, W \in \Gamma(TM)$,

$$\begin{aligned} d\eta(V,W) &= (\nabla_V\,\eta)(W) - (\nabla_W\,\eta)(V) \\ &= V(\eta(W)) - W(\eta(V)) - \eta([V,W]) = 0 \end{aligned} \tag{3.2}$$

This implies that T_zM is an isotropic subspace (see (2.1)) of the symplectic space $(H_x, d\eta_x)$, so the maximum dimension of T_zM is $n-1$ due to dimension theorem.

Remark. In the Legendrian submanifold $M \hookrightarrow S^{2n-1}$, if $\{\,\varepsilon_1, \ldots, \varepsilon_n\,\}$ forms the tangent frame field on TM, the normal frame field consists of $J\varepsilon_1, \ldots, J\varepsilon_n$ and the *Reeb vector field* $\vec{R}$.

3.2 Special Lagrangian Cone in $\mathbb{R}^{2n}$

Let M denote a minimal Legendrian submanifold in S^{2n-1}. Consider the cone CM over M, which is the image under the map

$$(x,t) \mapsto tx : M \times [0,\infty) \to \mathbb{R}^{2n} \cong \mathbb{C}^n. \tag{3.3}$$

Obviously CM has a singularity at $t = 0$, so one may introduce the *associated truncated cone* CM_ε, which is the image of $M \times [\varepsilon, \infty)$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ for all $\varepsilon > 0$ under the corresponding restriction of same map.

Our first result to be quoted here concerns on minimality (Simons [29]).

Proposition 3.1. *Let $\varepsilon > 0$. Then CM_ε is a minimal submanifold in $\mathbb{R}^{2n}$ if and only if M is a minimal submanifold in S^{2n-1}.*

Now fix a point $z \in M \subseteq S^{2n-1}$, we may choose a local orthonormal frame field e_s $(s = 1, \ldots, n-1)$ near $z \in M$. We can extend $e_s \in \Gamma(TM)$ to a local vector field $E_s \in \Gamma(T(CM))$ in CM by parallel translating along rays from $0 \in \mathbb{R}^{2n}$, i.e.

$$E_s = \frac{1}{r}\, e_s, \tag{3.4}$$

where $r := \sqrt{\sum_{j=1}^n (x_j^2 + y_j^2)}$ is the distance function from the origin.

Therefore, $\{ E_1, \ldots, E_{n-1}, \frac{\partial}{\partial r} \}$ is an orthonormal frame field in CM and $\frac{\partial}{\partial r}$ is the unit tangent vector field along rays. Also, since rays are geodesics,

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0, \tag{3.5}$$

where ∇ denotes the Euclidean connection on $\mathbb{R}^{2n}$.

We now state our second result.

Proposition 3.2. *M is a Legendrian submanifold in S^{2n-1} if and only if CM is a Lagrangian submanifold in $\mathbb{R}^{2n} \cong \mathbb{C}^n$.*

Proof. If M is Legendrian submanifold in S^{2n-1}, by definition and (3.2) one see that $\eta(e_s) = 0$ and $d\eta(e_s, e_t) = 0$ $(s, t = 1, \ldots, n-1)$. Now (3.1) implies that

$$\omega(E_s, E_t) = \frac{1}{r^2} \omega(e_s, e_t) = 0;$$

and $\omega(\frac{\partial}{\partial r}, E_s) = \langle J\frac{\partial}{\partial r}, E_s \rangle = \eta(E_s) = \frac{1}{r}\eta(e_s) = 0$. Therefore CM is a Lagrangian submanifold in $\mathbb{C}^n$. The converse is similar. $\square$

3.3 The Second Fundamental Form of Lagrangian Cone in $\mathbb{R}^{2n}$

Let M be a Legendrian submanifold in S^{2n-1}. In this section, we compute the coefficients of the second fundamental form of CM in $\mathbb{R}^{2n}$ in terms of those of M in S^{2n-1}, which can be used to compare the Gauss maps between them in Chapter 6.

Now $\{ E_s, \frac{\partial}{\partial r} \}$ $(s = 1, \ldots, n-1)$ is a local orthonormal frame field in CM. Thus being a Lagrangian submanifold, $\{ E_s, \frac{\partial}{\partial r}, JE_s, J\frac{\partial}{\partial r} \}$ is a local orthonormal frame field in $\mathbb{R}^{2n}$. The coefficients of the second fundamental form $\tilde{\mathrm{II}}$ of CM in $\mathbb{R}^{2n}$ we shall find are

$$\tilde{h}_{st}{}^{u} = \langle \nabla_{E_s} E_t, JE_u \rangle, \quad \tilde{h}_{st}{}^{n} = \left\langle \nabla_{E_s} E_t, J\frac{\partial}{\partial r} \right\rangle, \tag{3.6}$$

where $s, t, u = 1, \ldots, n-1$ and ∇ denotes the Euclidean connection on $\mathbb{C}^n$.

Since the second fundamental form is symmetric, $\tilde{h}_{st}{}^{u} \overset{1\leftrightarrow 2}{=} \tilde{h}_{ts}{}^{u}$; moreover,

$$\tilde{h}_{st}{}^{u} = \langle \nabla_{E_s} E_t, JE_u \rangle = -\langle E_t, \nabla_{E_s} JE_u \rangle = \langle \nabla_{E_s} E_u, JE_t \rangle \overset{2\leftrightarrow 3}{=} \tilde{h}_{su}{}^{t}.$$

Therefore, composition of the above implies $\tilde{h}_{st}{}^{u} \overset{1\leftrightarrow 3}{=} \tilde{h}_{ut}{}^{s}$. In addition, the same situation still applies for $\tilde{h}_{st}{}^{n}$, i.e. all indices are interchangeable. In general this property still holds for any Lagrangian submanifold.

Back to our situation, we compute that

$$\nabla_{E_s}\frac{\partial}{\partial r} = \nabla_{E_s}\frac{\vec{Z}}{r} = \frac{1}{r}\nabla_{E_s}\vec{Z} = \frac{1}{r}E_s, \tag{3.7}$$

where $\vec{Z}$ denotes the position vector field on $\mathbb{C}^n$

Furthermore from $0 = E_s\langle E_t, \frac{\partial}{\partial r}\rangle = \langle \nabla_{E_s}E_t, \frac{\partial}{\partial r}\rangle + \langle E_t, \nabla_{E_s}\frac{\partial}{\partial r}\rangle$, we get

$$\left\langle \nabla_{E_s}E_t, \frac{\partial}{\partial r}\right\rangle = -\left\langle E_t, \nabla_{E_s}\frac{\partial}{\partial r}\right\rangle = -\frac{1}{r}\langle E_t, E_s\rangle = -\frac{1}{r}\delta_{st} \tag{3.8}$$

To compute $\langle \nabla_{E_s}E_t, E_u\rangle$ and $\langle \nabla_{E_s}E_t, JE_u\rangle$, one can consider

$$\begin{aligned}\frac{d}{dr}\langle \nabla_{E_s}E_t, E_u\rangle &= \langle \nabla_{\frac{\partial}{\partial r}}\nabla_{E_s}E_t, E_u\rangle \quad (\because\ E_u \text{ is parallel, i.e. } \nabla_{\frac{\partial}{\partial r}}E_u = 0)\\ &= \langle \nabla_{E_s}\nabla_{\frac{\partial}{\partial r}}E_t, E_u\rangle + \langle \nabla_{[\frac{\partial}{\partial r}, E_s]}E_t, E_u\rangle \quad (\because\ \mathbb{R}^{2n} \text{ is flat})\\ &= \langle \nabla_{\nabla_{\partial/\partial r}E_s - \nabla_{E_s}\frac{\partial}{\partial r}}E_t, E_u\rangle \quad (\because\ \nabla \text{ is Levi-Civita})\\ &= -\frac{1}{r}\langle \nabla_{E_s}E_t, E_u\rangle \quad (\because\ (3.7));\end{aligned}$$

similarly, $\frac{d}{dr}\langle \nabla_{E_s}E_t, JE_u\rangle = -\frac{1}{r}\langle \nabla_{E_s}E_t, JE_u\rangle$. Thus by solving the ordinary differential equations, we get

$$\langle \nabla_{E_s}E_t, E_u\rangle = \frac{C_{st}{}^u}{r} \quad\text{and}\quad \langle \nabla_{E_s}E_t, JE_u\rangle = \frac{C'_{st}{}^u}{r}, \tag{3.9}$$

where $C_{st}{}^u$, $C'_{st}{}^u$ are constants along the ray. Now at $r = 1$, we have

$$C_{st}{}^u = \langle \nabla_{e_s}e_t, e_u\rangle = \langle \nabla^{\top}_{e_s}e_t, e_u\rangle = 0, \text{ and}$$

$$C'_{st}{}^u = \langle \nabla_{e_s}e_t, Je_u\rangle = \langle \mathrm{II}(e_s, e_t), Je_u\rangle = h_{st}{}^u,$$

where $\nabla^{\top}$ is the Levi-Civita connection on M and ${h_{st}}^u$ are the coefficients of the second fundamental form II of M in S^{2n-1} (§3.1 Remark) in the Je_u direction.

Therefore (3.9) now reads

$$\langle \nabla_{E_s} E_t, E_u \rangle = 0 \ \text{ and } \ {\tilde{h}_{st}}{}^u = \langle \nabla_{E_s} E_t, JE_u \rangle = \frac{{h_{st}}^u}{r} \tag{3.10}$$

We also need to compute by (3.7)

$$\begin{aligned} {\tilde{h}_{st}}{}^n &= {\tilde{h}_{sn}}{}^t = {\tilde{h}_{nt}}{}^s \\ &= \left\langle \nabla_{E_s} E_t, J\frac{\partial}{\partial r} \right\rangle = -\langle E_t, \nabla_{E_s} J\frac{\partial}{\partial r} \rangle = -\frac{1}{r} \langle E_t, JE_s \rangle = 0. \end{aligned} \tag{3.11}$$

In conclusion, by (3.10), (3.11) and (3.5), we have computed all the coefficients of the second fundamental form of CM in $\mathbb{R}^{2n}$. Writing them in the form of $n \times n$ matrix in each normal direction, we get

$$\tilde{h}^u = \begin{pmatrix} {h_{st}}^u / r & 0 \\ 0 & 0 \end{pmatrix}_{n \times n} \qquad \text{in the } JE_u \text{ direction,} \tag{3.12}$$

$$\tilde{h}^n = 0_{n \times n} \qquad \text{in the } J\frac{\partial}{\partial r} \text{ direction,} \tag{3.13}$$

where $s, t, u = 1, \ldots, n-1$.

Chapter 4

Geometry of Grassmannians

In this chapter, we shall give a brief survey on the results about Grassmann manifold $G(n,m)$ of oriented n-spaces in $\mathbb{R}^{n+m}$ instead of Grassmann manifold $G^*(n,m)$ of unoriented n-spaces in $\mathbb{R}^{n+m}$. Most of our results have been adapted from Jost and Xin [20], Hildebrandt, Jost and Widman [16], Leichtweiss [25], Wong [30, 31], and Fischer-Colbrie [11].

4.1 Locally Symmetric Space

Let (M^n, g) be a Riemannian manifold with metric $g = \langle \cdot\, , \cdot \rangle$, curvature tensor R and $x, x_0 \in M$. The manifold M is called a ***locally symmetric space*** if for any geodesic γ, $R(u,v)w$ is parallel along γ for any parallel vector field u, v, w along γ.

Let γ be a geodesic issuing from x_0 with $\gamma(0) = x_0$ and $\gamma(t) = x$, where t denotes the arc-length parameter. Let $R_{\dot\gamma(t)} : T_{\gamma(t)}M \to T_{\gamma(t)}M$ define a linear map

$$R_{\dot\gamma(t)} : v \mapsto R(v, \dot\gamma(t))\dot\gamma(t). \tag{4.1}$$

We can see that $R_{\dot\gamma}$ is self-adjoint on the space of vector fields along γ. In fact, for any vector fields v, w along γ,

$$\langle R_{\dot\gamma} v, w \rangle = \langle R(v, \dot\gamma)\dot\gamma, w \rangle \overset{12\leftrightarrow 34}{=} \langle R(\dot\gamma, w)v, \dot\gamma \rangle \underset{3\leftrightarrow 4}{\overset{1\leftrightarrow 2}{=}} \langle R(w, \dot\gamma)\dot\gamma, v \rangle = \langle v, R_{\dot\gamma} w \rangle,$$

where the second and third equalities is due to the symmetric properties of the curvature tensor. This means $R_{\dot\gamma}$ is diagonalizable and $T_{\gamma(t)}M$ admits eigenspace decomposition $\forall\ t$.

Let v_0 be a unit eigenvector of $R_{\dot\gamma(0)}$ with eigenvalue μ and $v_0 \perp \dot\gamma(0)$. We may extend it into a vector field $v(t)$ on M by parallel translation along γ. Suppose now M is a locally symmetric space with nonnegative sectional curvature, then $R(v, \dot\gamma)\dot\gamma$ is parallel and $R(v, \dot\gamma)\dot\gamma\,|_{t=0} = R_{\dot\gamma(0)}v_0 = \mu v_0$. Thus $R(v, \dot\gamma)\dot\gamma$ is the parallel translation of μv_0 and by uniqueness,

$$R(v, \dot\gamma)\dot\gamma = \mu v, \tag{4.2}$$

i.e. $v(t)$ is an eigenvector of $R_{\dot\gamma(t)}$ with eigenvalue $\mu = \langle R(v, \dot\gamma)\dot\gamma, v \rangle \geq 0$.

Consequently we can compute Jacobi fields J along γ vanishing at $t=0$. For if $\nabla^2_{\dot\gamma} J + R(J,\dot\gamma)\dot\gamma = 0$ where ∇ denotes the Levi-Civita connection on M, then $\nabla^2_{\dot\gamma} J = -R(J,\dot\gamma)\dot\gamma = -R_{\dot\gamma} J$. Thus,

$$J(t) = \begin{cases} \dfrac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} t)\, v(t), & \text{if } \mu > 0 \\ tv(t), & \text{if } \mu = 0. \end{cases} \tag{4.3}$$

Being a Jacobi field along γ, J can be treated as the variation field of some variation $\alpha(s,t) : (-\varepsilon, \varepsilon) \times [0,r] \to M$ of γ through geodesics. Therefore,

$$[J, \dot\gamma] = \left[\alpha_* \frac{\partial}{\partial s}, \alpha_* \frac{\partial}{\partial t}\right]\Bigg|_{s=0} = \alpha_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right]\Bigg|_{s=0} = 0, \tag{4.4}$$

where $[\,\cdot\,,\,\cdot\,]$ denotes the Lie bracket.

We may now assume γ is a geodesic without a conjugate point up to distance r from x_0. Therefore for any orthonormal vectors $w, \tilde w \in T_{\gamma(r)} B_r(x_0)$, we can extend them *uniquely* to Jacobi fields $J, \tilde J$ along γ vanishing at $t=0$, i.e. $J(0) = \tilde J(0) = 0$ and $J(r) = w$, $\tilde J(r) = \tilde w$.

By Gauss lemma, we have $\mathrm{Hess}(r)(w, \tilde w) = \langle \nabla_w \dot\gamma, \tilde w \rangle$ where $\frac{\partial}{\partial r} = \dot\gamma$ in a normal neighbourhood centred at x_0. Applying the extension from Jacobi fields, then

$$\mathrm{Hess}(r)(w, \tilde w) = \langle \nabla_w \dot\gamma, \tilde w \rangle = \langle \nabla_J \dot\gamma, \tilde J \rangle \Bigg|_0^r = \langle (\nabla_{\dot\gamma} J)(r), \tilde J(r) \rangle, \tag{4.5}$$

where the last equality is due to (4.4).

Let $v_i(t)$ denote orthonormal eigenvectors of $R_{\dot\gamma(t)}$ on $T_{\gamma(t)}B_r(x_0)$ with eigenvalues $\mu_i \geq 0$ (not necessarily distinct). Then $J_i(t)$ given by (4.3) are orthogonal Jacobi fields along γ. Therefore

$$\nabla_{\dot\gamma} J_i = \begin{cases} \cos(\sqrt{\mu_i}t)\, v_i(t), & \text{if } \mu_i > 0 \\ v_i(t), & \text{if } \mu_i = 0. \end{cases}$$

Consequently (4.5) implies

$$\begin{aligned}\operatorname{Hess}(r)\big(J_i(r), J_j(r)\big) &= \langle (\nabla_{\dot\gamma} J_i)(r), J_j(r) \rangle \\ &= \begin{cases} \left\langle \cos(\sqrt{\mu_i}r)\, v_i(r), \dfrac{1}{\sqrt{\mu_j}} \sin(\sqrt{\mu_j}r)\, v_j(r) \right\rangle, & \text{if } \mu_i, \mu_j > 0 \\ \langle v_i(r), r v_j(r) \rangle, & \text{if } \mu_i, \mu_j = 0 \end{cases} \\ &= \begin{cases} \dfrac{1}{\sqrt{\mu_j}} \sin(\sqrt{\mu_j}r) \cos(\sqrt{\mu_i}r)\, \delta_{ij}, & \text{if } \mu_i, \mu_j > 0 \\ r\, \delta_{ij}, & \text{if } \mu_i, \mu_j = 0 \end{cases}\end{aligned}$$

where $\mu_i = 0,\ \mu_j > 0 \ \Rightarrow\ i \neq j \ \Rightarrow\ v_i \perp v_j \ \Rightarrow\ \operatorname{Hess}\big(J_i(r), J_j(r)\big) = 0.$

On the other hand,

$$\operatorname{Hess}(r)\big(J_i(r), J_i(r)\big) = \begin{cases} \dfrac{1}{\mu_i} \sin(\sqrt{\mu_i}r) \cos(\sqrt{\mu_i}r) \operatorname{Hess}(r)\big(v_i(r), v_i(r)\big), & \text{if } \mu_i > 0 \\ r^2 \operatorname{Hess}(r)\big(v_i(r), v_i(r)\big), & \text{if } \mu_i = 0, \end{cases}$$

we have

$$\operatorname{Hess}(r)\big(v_i(r), v_j(r)\big) = \begin{cases} \sqrt{\mu_i} \cot(\sqrt{\mu_i}r)\, \delta_{ij}, & \text{if } \mu_i > 0 \\ \dfrac{1}{r}\, \delta_{ij}, & \text{if } \mu_i = 0 \end{cases} \tag{4.6}$$

constitutes the diagonal matrix representation of $\operatorname{Hess}(r)$.

4.2 The Grassmann manifold $G(n, m)$

Let $\mathbb{R}^{n+m}$ denote the real $(n+m)$-dimensional Euclidean space, with coordinate $x = (x_1, \ldots, x_{n+m})$. The set of all *oriented* n-planes P (i.e. n-dimensional oriented subspaces) constitute the ***Grassmann manifold*** or ***Grassmannian*** $G(n, m)$.

Fix an orthonormal basis $\{ e_1, \ldots, e_{n+m} \}$ for $\mathbb{R}^{n+m}$ and $P_0 := e_1 \wedge \cdots \wedge e_n$. In a neighbourhood $U_{12\cdots n} = U_{P_0}$ around P_0, we can span the n-planes P by n vectors f_α (not necessarily orthonormal), $\alpha = 1, \ldots, n$, given by

$$f_\alpha = e_\alpha + z_{\alpha i} e_{n+i}, \quad Z =: (z_{\alpha i}) \tag{4.7}$$

where i sums over from 1 to m, $P = f_1 \wedge \cdots \wedge f_n$ and vectors in P can be written in the form (x, xZ), $x \in \mathbb{R}^n$.

The expression (4.7) actually defines a local coordinate from the neighbourhood U_{P_0} onto $\mathbb{R}^{n \times m}$. The atlas of $G(n, m)$ can be defined to be $\{ U_{j_1 j_2 \cdots j_n} : 1 \leq j_1 \neq j_2 \cdots < j_n \leq n \}$, and $G(n, m)$ is thus an nm-dimensional manifold. It is known that $G(n, m)$ is compact and connected.

We can also regard $G(n, m)$ as a symmetric space and $G(n, m) \cong \mathrm{SO}(n + m) \,/\, \mathrm{SO}(n) \times \mathrm{SO}(n)$. There is an involution σ on $\mathrm{SO}(n+m)$ given by $\sigma : Q \mapsto$

SQS^{-1} where

$$S = \begin{pmatrix} -I_n & \\ & I_m \end{pmatrix}.$$

The subgroup $\mathrm{SO}(n) \times \mathrm{SO}(m)$ coincides with the identity component of the subgroup of all fixed elements of σ. It is customary that a symmetric space is in particular a locally symmetric space.

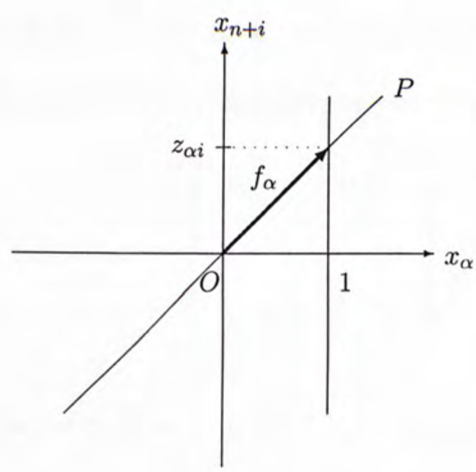

Figure 4.1: An illustration of the local coordinate on $G(n, m)$

Let P and Q be two points in $G(n, m)$. Wong [30] defined the ***Jordan angles*** between P and Q as the critical values θ_α of the angle between a nonzero vector x in P and its orthogonal projection $\tilde{x}$ in Q as x runs through P. The distance between P and Q can be defined by the square root of sum of the squares of the

n Jordan angles, i.e.

$$d(P,Q) := \sqrt{\theta_1^2 + \cdots + \theta_n^2} \tag{4.8}$$

and through tedious computation, the Riemannian metric on $G(n,m)$ in local coordinate Z (4.7) can be written as

$$ds^2 = \operatorname{tr}[(I_n + ZZ^T)^{-1} dZ (I_m + Z^T Z)^{-1} dZ^T]. \tag{4.9}$$

Let us assume that $n \le m$. Let T be a unit tangent vector at the point P_0 described by $Z = 0$. Without loss of generality, we may assume that

$$T = (\,\lambda_\alpha \, \delta_{\alpha i}\,) = \left(\begin{array}{ccc|c} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_n & \end{array}\right) \tag{4.10}$$

where $\displaystyle\sum_{\alpha=1}^{n} \lambda_\alpha^2 = 1$. Otherwise we may replace $e_1, \ldots, e_n$ and $e_{n+1}, \ldots, e_{n+m}$ by choosing suitable orthonormal bases for $e_1 \wedge \cdots \wedge e_n$ and $e_{n+1} \wedge \cdots \wedge e_{n+m}$.

We define a curve $Z = Z(t)$ in local coordinate (4.7) with the initial values $Z(0) = 0$ and $\dot{Z}(0) = T$ by

$$Z(t) = (\,\delta_{\alpha i} \tan \lambda_\alpha t\,) = \left(\begin{array}{ccc|c} \tan \lambda_1 t & & 0 & \\ & \ddots & & 0 \\ 0 & & \tan \lambda_n t & \end{array}\right) \tag{4.11}$$

for $0 \le t < \frac{\pi}{2|\lambda_{\alpha'}|}$, where $|\lambda_{\alpha'}| := \max\{|\lambda_1|, \ldots, |\lambda_n|\}$ and t denotes the parameter of arc length. $Z(t)$ can be shown to be a minimizing curve and in particular a geodesic, which records the rotation of n-planes in $G(n, m)$.

4.3 Leichtweiss' Formula for Curvature Tensor in $G(n, m)$

We cite the following result by Leichtweiss [25, §3.1] which describe the curvature tensor in $G(n, m)$ at a prescribed point $P_0 \in G(n, m)$.

Theorem 4.1. *Under the metric* (4.9), *the curvature tensor of* $G(n, m)$ *is*

$$R_{(\alpha i)(\beta j)(\gamma k)(\delta l)}|_{Z=0} = \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{il}\delta_{kj} + \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{ij}\delta_{kl} - \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ik}\delta_{jl} - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{kl}$$

in a local frame field $\{E_{\alpha i}\}$ *where* $E_{\alpha i}$ *denotes the* $n \times m$ *matrix of which the* (α, i)*-entry is* 1 *and* 0 *elsewhere.*

We shall present a different proof from what Leichtweiss has shown. Before that, for $P \in U_{P_0}$ we may identify $T_P G(n, m) \cong \mathbb{R}^{n \times m}$. Therefore,

$$\langle X, Y \rangle_P = \operatorname{tr}[(I + ZZ^T)^{-1} X (I + Z^T Z)^{-1} Y^T] \tag{4.12}$$

where $X, Y \in \mathbb{R}^{n \times m}$ are treated as vectors at P in U_{P_0}.

We let $E_{\alpha i}$ be so defined in Theorem 4.1. Denote the metric coefficients by

$$g_{(\alpha i)(\beta j)} := \langle\, E_{\alpha i}, E_{\beta j}\,\rangle,$$

implying that

$$g_{(\alpha i)(\beta j)}|_{Z=0} = \mathrm{tr}\,(E_{\alpha i}E_{\beta j}^T) = \delta_{\alpha\beta}\delta_{ij}, \tag{4.13}$$

and its inverse $g^{(\alpha i)(\beta j)}$. Denote by ∇ the Levi-Civita connection with respect to the metric (4.9) on $G(n,m)$, which also satisfies

$$\nabla_{E_{\alpha i}} E_{\beta j} = \Gamma^{\gamma k}_{(\alpha i)(\beta j)} E_{\gamma k}.$$

To find the Christoffel symbols $\Gamma^{\gamma k}_{(\alpha i)(\beta j)}$ at P_0, we can compute that

$$\begin{aligned} &W\,\langle\, X, Y\,\rangle \\ &\quad= -\mathrm{tr}\,[\,(I+ZZ^T)^{-1}(WZ^T+ZW^T)(I+ZZ^T)^{-1}X(I+Z^TZ)^{-1}Y^T \\ &\qquad\quad + (I+ZZ^T)^{-1}X(I+Z^TZ)^{-1}(W^TZ+Z^TW)(I+Z^TZ)^{-1}Y^T\,] \qquad (4.14) \\ &\quad= -\mathrm{tr}\,[\,(I+ZZ^T)^{-1}(WZ^T+ZW^T)(I+ZZ^T)^{-1}X(I+Z^TZ)^{-1}Y^T\,] \\ &\qquad - \mathrm{tr}\,[\,(I+Z^TZ)^{-1}(W^TZ+Z^TW)(I+Z^TZ)^{-1}Y^T(I+ZZ^T)^{-1}X\,]. \end{aligned}$$

Now we find that for $\alpha, \beta, \cdots = 1, \ldots, n$ and $i, j, \cdots = 1, \ldots, m$,

$$E_{\alpha i}\langle\, E_{\beta j}, E_{\delta l}\,\rangle|_{Z=0} = -\mathrm{tr}\,0 = 0.$$

A classical formula shows that at P_0,

$$\begin{aligned} &\Gamma^{\gamma k}_{(\alpha i)(\beta j)}\Big|_{Z=0} \\ &\quad= \frac{1}{2}\, g^{(\gamma k)(\delta l)}(E_{\alpha i}\,\langle\, E_{\beta j}, E_{\delta l}\,\rangle + E_{\beta j}\,\langle\, E_{\delta l}, E_{\alpha i}\,\rangle - E_{\delta l}\,\langle\, E_{\alpha i}, E_{\beta j}\,\rangle)\Big|_{Z=0} = 0. \end{aligned} \tag{4.15}$$

Due to (4.13), (4.15), we can apply a classical formula [24, p.128 Problem 7-1] for computing $R_{(\alpha i)(\beta j)(\gamma k)(\delta l)}$ locally at P_0. That is,

$$
\begin{aligned}
& R_{(\alpha i)(\beta j)(\gamma k)(\delta l)}|_{Z=0} \\
&\quad = \frac{1}{2}\big(E_{\beta j}E_{\delta l}\langle E_{\alpha i}, E_{\gamma k}\rangle + E_{\alpha i}E_{\gamma k}\langle E_{\beta j}, E_{\delta l}\rangle \\
&\qquad\qquad - E_{\beta j}E_{\gamma k}\langle E_{\alpha i}, E_{\delta l}\rangle - E_{\alpha i}E_{\delta l}\langle E_{\beta j}, E_{\gamma k}\rangle\big)\Big|_{Z=0}.
\end{aligned}
\tag{4.16}
$$

Therefore we need to compute the second derivatives $VW\langle X, Y\rangle$ for any vectors X, Y, W, V at P, i.e. we need to differentiate (4.14) one more time. Hence we now write

$$
\begin{aligned}
&[I + (Z + tV)(Z + tV)^T]^{-1}[W(Z + tV)^T + (Z + tV)W^T] \\
&\qquad [I + (Z + tV)(Z + tV)^T]^{-1}X[I + (Z + tV)^T(Z + tV)]^{-1}Y^T \\
&= (I + ZZ^T)^{-1}(WZ^T + ZW^T)(I + ZZ^T)^{-1}X(I + Z^TZ)^{-1}Y^T \\
&\quad + t\,\big[(I + ZZ^T)^{-1}(WV^T + VW^T)(I + ZZ^T)^{-1}X(I + Z^TZ)^{-1}Y^T \\
&\qquad - (I + ZZ^T)^{-1}(WZ^T + ZW^T) \\
&\qquad\quad \big((I + ZZ^T)^{-1}(WZ^T + ZW^T)(I + ZZ^T)^{-1}X(I + Z^TZ)^{-1}Y^T \\
&\qquad\quad + (I + ZZ^T)^{-1}X(I + Z^TZ)^{-1}(W^TZ + Z^TW)(I + Z^TZ)^{-1}Y^T\big)\big] + O(t^2)
\end{aligned}
$$

and the second term in (4.14) can be differentiated by replacing Z with Z^T, W with W^T, X with Y^T, and Y with X^T.

Therefore locally at P_0, we found

$$\begin{aligned}
&E_{\beta j}E_{\delta l}\langle\, E_{\alpha i}, E_{\gamma k}\,\rangle|_{Z=0}\\
&\qquad= -\operatorname{tr}(E_{\delta l}E_{\beta j}^T + E_{\beta j}E_{\delta l}^T)E_{\alpha i}E_{\gamma k}^T - \operatorname{tr}(E_{\delta l}^T E_{\beta j} + E_{\beta j}^T E_{\delta l})E_{\gamma k}^T E_{\alpha i}\\
&\qquad= -\,\delta_{jl}\delta_{\alpha\beta}\delta_{ik}\delta_{\gamma\delta} - \delta_{jl}\delta_{\alpha\delta}\delta_{ik}\delta_{\beta\gamma} - \delta_{\beta\delta}\delta_{jk}\delta_{\alpha\gamma}\delta_{il} - \delta_{\beta\delta}\delta_{kl}\delta_{\alpha\gamma}\delta_{ij}.
\end{aligned}$$

By permuting the indices of the equation

$$E_{\beta j}E_{\delta l}\langle\, E_{\alpha i}, E_{\gamma k}\,\rangle|_{Z=0} = -\,\delta_{jl}\delta_{\alpha\beta}\delta_{ik}\delta_{\gamma\delta} - \delta_{jl}\delta_{\alpha\delta}\delta_{ik}\delta_{\beta\gamma} - \delta_{\beta\delta}\delta_{jk}\delta_{\alpha\gamma}\delta_{il} - \delta_{\beta\delta}\delta_{kl}\delta_{\alpha\gamma}\delta_{ij}$$

accordingly, and employing the formula (4.16), one can get

$$\begin{aligned}
&R_{(\alpha i)(\beta j)(\gamma k)(\delta l)}|_{Z=0}\\
&\qquad= -\,\delta_{jl}\delta_{\alpha\beta}\delta_{ik}\delta_{\gamma\delta} - \delta_{jl}\delta_{\alpha\delta}\delta_{ik}\delta_{\beta\gamma} - \delta_{\beta\delta}\delta_{jk}\delta_{\alpha\gamma}\delta_{il} - \delta_{\beta\delta}\delta_{kl}\delta_{\alpha\gamma}\delta_{ij}\\
&\qquad\quad + \delta_{jk}\delta_{\alpha\beta}\delta_{il}\delta_{\gamma\delta} + \delta_{jk}\delta_{\alpha\gamma}\delta_{il}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{jl}\delta_{\alpha\delta}\delta_{ik} + \delta_{\beta\gamma}\delta_{kl}\delta_{\alpha\delta}\delta_{ij} \qquad (4.17)\\
&\qquad= \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{il}\delta_{kj} + \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{ij}\delta_{kl} - \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ik}\delta_{jl} - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{kl}.
\end{aligned}$$

That is to say, Theorem 4.1 is proved.

4.4 Normal Neighbourhoods of a Point in $G(n, m)$

In the previous sections, we have computed the Jacobi fields (4.3) and the eigenvalues (4.6) of the Hessian of the distance function r on $G(n,m)$ from a given fixed point $P_0 \in G(n,m)$ in terms of the eigenvalues of the adjoint operator $R_{\dot\gamma}$ (4.1) at P_0, since $G(n,m)$ is a symmetric space. We now let $\dot\gamma = x_{\alpha i}\,E_{\alpha i}$ and

$v = v_{\alpha i}\, E_{\alpha i}$ be vectors at P_0, then by (4.17),

$$\begin{aligned}
\langle\, R(E_{\alpha i}, \dot\gamma)\dot\gamma, v\,\rangle|_{Z=0} &= x_{\beta j}x_{\gamma k}v_{\delta l}\ \langle\, R(E_{\alpha i}, E_{\beta j})E_{\gamma k}, E_{\delta l}\,\rangle|_{Z=0} \\
&= x_{\beta j}x_{\gamma k}v_{\delta l}\,(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{il}\delta_{kj} + \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{ij}\delta_{kl} \\
&\qquad - \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ik}\delta_{jl} - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{kl}) \\
&= x_{\alpha j}x_{\gamma j}v_{\gamma i} + x_{\gamma i}x_{\gamma k}v_{\alpha k} - x_{\alpha j}x_{\gamma i}v_{\gamma j} - x_{\beta i}x_{\alpha k}v_{\beta k} \\
&= x_{\alpha j}x_{\beta j}v_{\beta i} + x_{\beta i}x_{\beta j}v_{\alpha j} - 2\, x_{\alpha j}x_{\beta i}v_{\beta j}.
\end{aligned} \tag{4.18}$$

Without loss of generality, we may assume $n \le m$, and that the tangent vector $\dot\gamma$ at P_0, by an action of $\mathrm{SO}(m) \times \mathrm{SO}(n)$, can be given by

$$X := (\, x_{\alpha i}\,) = (\, \lambda_\alpha \delta_{\alpha i}\,)$$

where $\sum_\alpha \lambda_\alpha^2 = 1$. By simple linear algebra, there exist an $n \times n$ orthogonal matrix U and an $m \times m$ matrix $\tilde U$ such that

$$UX\tilde U = (\, \lambda_\alpha \delta_{\alpha i}\,).$$

Thanks to (4.18) and (4.13), the adjoint operator $R_{\dot\gamma}$ (4.1) at P_0 is written as

$$\begin{aligned}
R_{\dot\gamma} v &= (\lambda_\alpha\lambda_\beta\delta_{\alpha j}\delta_{\beta j}v_{\beta i} + \lambda_\beta^2\delta_{\beta i}\delta_{\beta j}v_{\alpha j} - 2\,\lambda_\alpha\lambda_\beta\delta_{\alpha j}\delta_{\beta i}v_{\beta j})E_{\alpha i} \\
&= (\lambda_\alpha^2 v_{\alpha i} + \lambda_\beta^2\delta_{\beta i}v_{\alpha\beta} - 2\,\lambda_\alpha\lambda_\beta\delta_{\beta i}v_{\beta\alpha})E_{\alpha i} \\
&= \begin{cases} (\lambda_\alpha^2 v_{\alpha\beta} + \lambda_\beta^2 v_{\alpha\beta} - 2\,\lambda_\alpha\lambda_\beta v_{\beta\alpha})E_{\alpha\beta} & \text{if } i = \beta = 1, \ldots, n \\ \lambda_\alpha^2 v_{\alpha s}E_{\alpha s} & \text{if } i = s = n+1, \ldots, m. \end{cases}
\end{aligned} \tag{4.19}$$

We can always decompose the vector space $\mathbb{R}^{n\times m}$ into a direct sum of a space of $n\times n$ symmetric matrices, a space of $n\times n$ skew-symmetric matrices and $\mathbb{R}^{n\times(m-n)}$.

We adopt a custom $\alpha, \beta = 1, \ldots, n$ and $s = n+1, \ldots, m$. In the case $v_{\alpha\beta} = v_{\beta\alpha}$, we found from (4.19) that

$$\begin{aligned} R_{\dot\gamma}(v_{\alpha\beta}E_{\alpha\beta}) &= (\lambda_\alpha^2 - 2\,\lambda_\alpha\lambda_\beta + \lambda_\beta^2)v_{\alpha\beta}E_{\alpha\beta} \\ &= (\lambda_\alpha - \lambda_\beta)^2(v_{\alpha\beta}E_{\alpha\beta}). \end{aligned}$$

In the case $v_{\alpha\beta} = -v_{\beta\alpha}$,

$$\begin{aligned} R_{\dot\gamma}(v_{\alpha\beta}E_{\alpha\beta}) &= (\lambda_\alpha^2 + 2\,\lambda_\alpha\lambda_\beta + \lambda_\beta^2)v_{\alpha\beta}E_{\alpha\beta} \\ &= (\lambda_\alpha + \lambda_\beta)^2(v_{\alpha\beta}E_{\alpha\beta}). \end{aligned}$$

Together that $R_{\dot\gamma}(v_{\alpha s}E_{\alpha s}) = \lambda_\alpha^2(v_{\alpha s}E_{\alpha s})$ in (4.19), we have diagonalized $R_{\dot\gamma}$ and form the Table 4.1.

Moreover by putting $\mu_i := \lambda_\alpha^2, (\lambda_\alpha \pm \lambda_\beta)^2, 0 \geq 0$ into (4.6), we can find the eigenvalues of the Hessian of the distance function r from P_0 at the direction $\dot\gamma = (\,\lambda_\alpha\delta_{\alpha i}\,)$ which we tabulate them in Table 4.2.

Eigenvalues	Basis eigenvectors	Number of basis eigenvectors	No. of similar eigenvalues
λ_α^2	$E_{\alpha\, n+1}, \ldots, E_{\alpha m}$	$m-n$	n
$(\lambda_\alpha+\lambda_\beta)^2$	$E_{\alpha\beta}-E_{\beta\alpha}$	1	$\frac{n(n-1)}{2}$
$(\lambda_\alpha-\lambda_\beta)^2$	$E_{\alpha\beta}+E_{\beta\alpha}$	1	$\frac{n(n-1)}{2}$
0	$E_{11}, \ldots, E_{nn}$	n	1

Table 4.1: Eigenvalues of $R_{\dot\gamma}$ at P_0 $(\alpha \neq \beta)$

μ_i	$\sqrt{\mu_i}$	Eigenvalues of Hess(r)
λ_α^2	$\lvert\lambda_\alpha\rvert$	$\lambda_\alpha \cot \lambda_\alpha r$
$(\lambda_\alpha+\lambda_\beta)^2$	$\lvert\lambda_\alpha+\lambda_\beta\rvert$	$(\lambda_\alpha+\lambda_\beta)\cot(\lambda_\alpha+\lambda_\beta)r$
$(\lambda_\alpha-\lambda_\beta)^2$	$\lvert\lambda_\alpha-\lambda_\beta\rvert$	$(\lambda_\alpha-\lambda_\beta)\cot(\lambda_\alpha-\lambda_\beta)r$
0	0	$\frac{1}{r}$

Table 4.2: Eigenvalues of Hess(r) $(\alpha \neq \beta)$

Thus the Hessian of the distance function r from P_0 hence that of the square of the distance function r, i.e. Hess(r^2), remains positive definite for

$$r < \frac{\pi}{2 \max \sqrt{\mu_i}}. \tag{4.20}$$

Moreover by (4.3), there are no conjugate points of $P_0 \in G(n,m)$ along γ up to

$$t < \frac{\pi}{\max \sqrt{\mu_i}}, \tag{4.21}$$

where from Table 4.2 $\sqrt{\mu_i} \leq |\lambda_{\alpha'}| + |\lambda_{\beta'}|$ for all i, and $\lambda_{\alpha'}$ and $\lambda_{\beta'}$ are the two with the largest and the second largest absolute values among all λ_α's in the unit tangent vector $\dot{\gamma}(0) = (\lambda_\alpha \delta_{\alpha i})$ at P_0 respectively.

Let us define an open set $B_G(P_0)$ in $U_{P_0} \subseteq G(n,m)$. In the patch U_{P_0}, we have the local coordinate (4.7) around P_0, and then we will have the *normal polar coordinate* around P_0. Define $B_G(P_0)$, suggested by Jost and Xin [20, p.287], in normal polar coordinate around P_0 as

$$B_G(P_0) := \left\{ (X,t) : X = (\lambda_\alpha \delta_{\alpha i}), 0 \leq t < t_X := \frac{\pi}{2(|\lambda_{\alpha'}| + |\lambda_{\beta'}|)} \right\} \tag{4.22}$$

where $\lambda_{\alpha'}$ and $\lambda_{\beta'}$ are defined similarly as above. From the formula (4.11), we see that $B_G(P_0)$ lies inside the cut-locus of P_0. Furthermore we have learned from Table 4.2 that the square of the distance function r^2 from P_0 is a strictly convex smooth function in $B_G(P_0)$.

We would like to show that the open set $B_G(P_0)$ is indeed geodesically convex. To see that, let $P := ((\lambda_\alpha \delta_{\alpha i}), t)$ and $Q := ((\eta_\alpha \delta_{\alpha i}), s)$ be two points in $B_G(P_0)$. Then in terms of local coordinate (4.7), we have that in U_{P_0},

$$P = (\delta_{\alpha i} \tan \lambda_\alpha t) \quad \text{and} \quad Q = (\delta_{\alpha i} \tan \eta_\alpha s).$$

Consider a curve $\zeta = \zeta(h)$ between P and Q defined by

$$\zeta(h) := \left(\delta_{\alpha i} \tan\left((1-h)\lambda_\alpha t + h\eta_\alpha s\right) \right)$$

in the patch U_{P_0} and $h \in [0,1]$ is the parameter. We claim that ζ is a geodesic.

To show this, let $\bar{P}$ be the *mid-point* between P and Q defined by

$$\bar{P} = \left(\delta_{\alpha i} \tan \frac{\lambda_\alpha t + \eta_\alpha s}{2} \right)$$

in the patch U_{P_0}. Therefore by taking inner product, we may obtain the Jordan angles between P and $\bar{P}$ are $\frac{1}{2} \,|\, \lambda_\alpha t - \eta_\alpha s \,|$ where each angle

$$\frac{1}{2} \,|\, \lambda_\alpha t - \eta_\alpha s \,| \le \frac{1}{2} \left(\,|\, \lambda_\alpha t \,| + |\, \eta_\alpha s \,| \,\right) < \frac{\pi}{2},$$

and the distance between P and $\bar{P}$ is

$$d(P, \bar{P}) = \frac{1}{2} \sqrt{\sum_\alpha (\, \lambda_\alpha t - \eta_\alpha s \,)^2}. \tag{4.23}$$

Let $\tilde{P}$ be an arbitrary point between P and $\bar{P}$ on ζ, i.e.

$$\tilde{P} = \left(\delta_{\alpha i} \tan\left((1-h)\lambda_\alpha t + h\eta_\alpha s\right) \right)$$

and $0 < h < \frac{1}{2}$. By similar computation before, we found

$$d(P, \tilde{P}) = h \sqrt{\sum_\alpha (\lambda_\alpha t - \eta_\alpha s)^2} \quad \text{and} \quad d(\tilde{P}, \bar{P}) = \left(\frac{1}{2} - h \right) \sqrt{\sum_\alpha (\lambda_\alpha t - \eta_\alpha s)^2}.$$

This shows that the distance function d is additive along the curve $\zeta|_{[0,\frac{1}{2}]}$, thus the segment $\zeta|_{[0,\frac{1}{2}]}$ between P and $\bar{P}$ is a minimizing curve.

Repeating the same argument with suitable pairs of P and Q, we can conclude every other similar segment of ζ is also minimizing. Therefore the whole curve ζ is a geodesic between P and Q.

We claim also that $\zeta \subseteq B_G(P_0)$ and $\zeta \subseteq B_G(\bar{P})$. Now we note that for any $h \in [0,1]$, $\zeta(h)$ is a point on a geodesic starting from P_0 at the direction

$$\tilde{X} = \left(\frac{1}{A} \left((1-h)\lambda_\alpha t + h\eta_\alpha s \right) \delta_{\alpha i} \right)$$

where $A := \sum_\alpha \left((1-h)\lambda_\alpha t + h\eta_\alpha s \right)^2 > 0$. By our construction of $B_G(P_0)$ the radius in this direction is

$$t_{\tilde{X}} := \frac{\pi A}{2\left(|\,(1-h)\lambda_1 t + h\eta_1 s\,| + |\,(1-h)\lambda_2 t + h\eta_2 s\,| \right)}$$

where, without loss of generality, the first two components have the largest absolute values. Since $P, Q \in B_G(P_0)$ and $t < \frac{\pi}{2(|\lambda_{\alpha'}|+|\lambda_{\beta'}|)}$ and $s < \frac{\pi}{2(|\eta_{\gamma'}|+|\eta_{\delta'}|)}$,

$$\begin{aligned} &|\,(1-h)\lambda_1 + h\eta_1 s\,| + |\,(1-h)\lambda_2 + h\eta_2 s\,| \\ &\qquad \leq (1-h)(|\,\lambda_1\,| + |\,\lambda_2\,|)t + h(|\,\eta_1\,| + |\,\eta_2\,|)s \\ &\qquad < (1-h)\frac{(|\,\lambda_1\,| + |\,\lambda_2\,|)\pi}{2(|\,\lambda_{\alpha'}\,| + |\,\lambda_{\beta'}\,|)} + h\frac{(|\,\eta_1\,| + |\,\eta_2\,|)\pi}{2(|\,\eta_{\gamma'}\,| + |\,\eta_{\delta'}\,|)} \leq \frac{\pi}{2} \end{aligned}$$

hence $t_{\tilde{X}} > A$, which means $\zeta(h) \in B_G(P_0)\ \forall\ h$ and we proved the first claim.

Now we have to also show $\zeta \subseteq B_G(\bar{P})$. This can be done by change of bases. The n-plane $\bar{P}$ is spanned by n orthonormal vectors

$$\bar{f}_\alpha = \cos\frac{\lambda_\alpha t + \eta_\alpha s}{2}\, e_\alpha + \sin\frac{\lambda_\alpha t + \eta_\alpha s}{2}\, e_{n+\alpha},$$

which are complemented by

$$\bar{f}_{n+\alpha} = -\sin\frac{\lambda_\alpha t + \eta_\alpha s}{2}\, e_\alpha + \cos\frac{\lambda_\alpha t + \eta_\alpha s}{2}\, e_{n+\alpha}$$

and the remaining $m - n$ vectors do not change. Thus $\{\bar{f}_\alpha, \bar{f}_{n+\alpha}\}$ forms an orthonormal bases for $\mathbb{R}^{n+m}$. Therefore, each point $\zeta(h)$ on the geodesic ζ is an n-plane in $\mathbb{R}^{n+m}$ spanned by

$$\begin{aligned}\tilde{f}_\alpha &= \cos\big((1-h)\lambda_\alpha t + h\eta_\alpha s\big)\, e_\alpha + \sin\big((1-h)\lambda_\alpha t + h\eta_\alpha s\big)\, e_{n+\alpha} \\ &= \cos\!\left(\left(h - \frac{1}{2}\right)(\eta_\alpha t - \lambda_\alpha s)\right)\bar{f}_\alpha + \sin\!\left(\left(h - \frac{1}{2}\right)(\eta_\alpha t - \lambda_\alpha s)\right)\bar{f}_{n+\alpha}\end{aligned}$$

which means that the *shifted* geodesic ζ in coordinate chart $U^+_{\bar{P}}$ around $\bar{P}$ can be described by

$$\left(\tan\!\left(\left(h - \frac{1}{2}\right)(\eta_\alpha t - \lambda_\alpha s)\right)\delta_{\alpha i}\right)$$

where the tangent direction at $\bar{P}$ is $\big((\eta_\alpha t - \lambda_\alpha s)\delta_{\alpha i}\big)$. By argument similar to what we have used to proved $\zeta \subseteq B_G(P_0)$, we can show $\zeta \subseteq B_G(\bar{P})$.

Moreover again by change of bases, one finds that the geodesic ζ in coordinate chart U_P around P by $(\tan(h(\eta_\alpha t - \lambda_\alpha s))\,\delta_{\alpha i})$ where the tangent vector of γ at P is $((\eta_\alpha t - \lambda_\alpha s)\delta_{\alpha i})$, and we denote by $\bar{X}$ its unit direction. Therefore, we conclude from the above two claims that any geodesic ζ emanating from P to Q must lie in $B_G(P_0)$ and have length less than $2t_{\bar{X}}$. Owing to (4.20) and (4.21), we know that the geodesic ζ from P with length $< 2t_{\bar{X}}$ has no conjugate points and the square of distance function r^2 from P_0 remains strictly convex along ζ up to $t < t_X$ where t is the arc-length parameter.

Theorem 4.2. *In $B_G(P_0)$ the square of the distance function from P_0 is a smooth strictly convex function. Furthermore, $B_G(P_0)$ is a convex set, namely any two points in $B_G(P_0)$ can be joined in $B_G(P_0)$ by a* ***unique*** *geodesic arc. This arc does not contain a pair of conjugate points.*

Proof. Define $X := \{(P,Q) \in B_G(P_0) \times B_G(P_0) : P$ and Q can be joined by 2 geodesics.$\}$, which is bounded and closed for otherwise two geodesics would converge to one geodesic which admits a nonzero Jacobi field vanishes at P and Q, i.e. P is conjugate to Q, thus contradiction arises.

Assume $X \neq \varnothing$, then there is a pair $(P,Q) \in X$ joining by two minimal geodesics γ_1 and γ_2, thus, cf. [14], γ_1 and γ_2 form a closed geodesic $\gamma : S^1 \to B_G(P_0)$. Moreover, the distance function r^2 from P_0 is strictly convex on $B_G(P_0)$, by Corollary 5.1 γ is a constant map, leading to a contradiction. Thus $X = \varnothing$. $\square$

Wong [31] has given an upper bound for the sectional curvature of $G(n, m)$, i.e.

Lemma 4.1. *The sectional curvature of the Grassmann manifold $G(n, m)$ is less than or equal to 2 if $\min(n, m) \geq 2$ and less than or equal to 1 if $\min(n, m) = 1$.*

Proof. With reference to Table 4.2, we see that all sectional curvatures μ should be the convex combinations of λ_α^2, $(\lambda_\alpha \pm \lambda_\beta)^2$ and 0, where $\sum_\alpha \lambda_\alpha^2 = 1$. Therefore,

$$\mu \leq \max\{\, \lambda_\alpha^2, (\lambda_\alpha \pm \lambda_\beta)^2 \,\} \leq \max\left\{ 1, 4\left(\frac{\lambda_\alpha \pm \lambda_\beta}{2}\right)^2 \right\}$$
$$\leq \max\left\{ 1, 4 \cdot \frac{\lambda_\alpha^2 + \lambda_\beta^2}{2} \right\} \leq 2.$$

Furthermore from Table 4.1, if $\min(n, m) = 1$, all sectional curvatures can only be the convex combinations of λ_α^2 and 0, so they are less than or equal to 1. $\square$

It is now natural to introduce the ***usual convex geodesic ball*** $B_R(P_0)$ with the radius $R < \frac{\pi}{2\sqrt{2}}$ if $\min\{n, m\} \geq 2$ or $R < \frac{\pi}{2}$ if $\min(n, m) = 1$. By virtue of (4.22), we know $B_G(P_0) \subseteq B_R(P_0)$.

4.5 Some Remarks on Lagrangian Grassmannians

In §2.1, we have briefly introduced that the Lagrangian Grassmannian $\mathrm{Lag}(n) \cong \mathrm{U}(n) / \mathrm{SO}(n)$, which is thus a symmetric space. Clearly $\mathrm{Lag}(n) \subseteq G(n, n)$.

In terms of local coordinate (4.7) on $G(n, n)$, we see that for any $P \in \mathrm{Lag}(n) \subseteq G(n, n)$, we may let $u := (x, xZ)$ and $\tilde{u} := (\tilde{x}, \tilde{x}Z)$ be any two vectors on P. By definition of Lagrangian planes,

$$\langle u, J\tilde{u} \rangle = 0$$

where $Ju = \tilde{x}\,(\,-Z\ I\,)$ defines the complex structure of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Now

$$0 = \langle u, J\tilde{u} \rangle = x\,(\,I\ Z\,)\begin{pmatrix} -Z^T \\ I \end{pmatrix}\tilde{x}^T = -xZ^T\tilde{x}^T + xZ\tilde{x}^T$$

This implies $Z^T = Z$, that is, the Lagrangian Grassmannian $\mathrm{Lag}(n)$ is a connected component of the fixed point set of the transpose $Z \mapsto Z^T$ on $G(n, n)$. By observing the metric (4.9), one could see that the transpose $Z \mapsto Z^T$ is an isometry of $G(n, n)$. By a classical result of Klingenberg [23, 1.10.15 Theorem], the Lagrangian Grassmannian $\mathrm{Lag}(n)$ is a totally geodesic submanifold in $G(n, n)$.

By Gauss equation, we can see that the curvature tensor of $\mathrm{Lag}(n)$ at a given point $P_0 \in \mathrm{Lag}(n)$ can also be given by the formula (4.17).

Of course the geodesic $\gamma = \gamma(t)$ from P_0 at $\dot{\gamma}(0) = (\lambda_\alpha \delta_{\alpha i})$ by an action of $\mathrm{SO}(n)$, in local coordinate (4.7) on the patch U_{P_0} with $Z = Z^T$ is still given by Wong's formula (4.11).

In our situation, since the tangent vectors are all symmetric matrices and $m = n$, the eigenvalues of the adjoint operator $R_{\dot{\gamma}}$ and those of the Hessian of the distance function r (sufficiently small) from P_0 at the direction $\dot{\gamma} = (\lambda_\alpha \delta_{\alpha i})$ can now be found from the third and fourth rows in Tables 4.1 and 4.2.

Chapter 5

Harmonic Maps between Riemannian Manifolds

In this chapter, we review the concept of harmonic maps between Riemannian manifolds. Results on this issues are taken from Eells and Lemaire [9] and Eells and Sampson [10]. Then we would study the Gauss map and its tension field from Ruh and Vilms [28]. Finally we would introduce the notion of simple Riemannian manifold and quote a Liouville-type result by Hildebrandt, Jost and Widman [16].

5.1 Energy Functional and Tension Field

Let (M, g) and $(\tilde{M}, \tilde{g})$ be n-dimensional and m-dimensional Riemannian manifolds. For $\phi \in C^\infty(M, \tilde{M})$, there is a *linear map* $\phi_*|_x : T_x M \to T_{\phi(x)}\tilde{M}$, i.e.

$$\phi_*|_x \in \mathrm{Hom}(T_x M, T_{\phi(x)}\tilde{M}) \cong T_x^* M \otimes T_{\phi(x)}\tilde{M}.$$

The space of $(1,1)$-tensors $T_x^* M \otimes T_{\phi(x)}\tilde{M}$ has the standard metric coefficients $g^{\alpha\beta}(x)\tilde{g}_{ij}(u(x))$ for $\alpha, \beta = 1, \dots, n$ and $i, j = 1, \dots, n$.

In terms of local coordinates (x_α) and (y_i) on M and $\tilde{M}$ respectively, $\phi_*|_x$ is represented by

$$\phi_*|_x = \frac{\partial \phi^i}{\partial x_\alpha}(x)(dx^\alpha)_x \otimes \left(\frac{\partial}{\partial y_i}\right)_{\phi(x)}.$$

Therefore, we may write

$$|\,\phi_*\,|^2(x) := \langle\, \phi_*|_x, \phi_*|_x \,\rangle = g^{\alpha\beta}(x)\frac{\partial \phi^i}{\partial x_\alpha}(x)\frac{\partial \phi^j}{\partial x_\beta}(x)\tilde{g}_{ij}(\phi(x)) \tag{5.1}$$

which is also called the ***Hilbert-Schmidt norm***. Therefore,

Definition 5.1. *Given $\phi \in C^\infty(M, \tilde{M})$, the* ***energy density*** *of ϕ is the function $e(\phi) : M \to [0, \infty)$ defined as*

$$e(\phi)(x) := \frac{1}{2}\,|\,\phi_*\,|^2(x), \quad x \in M \tag{5.2}$$

where $|\cdot|$ denotes the Hilbert-Schmidt norm on $T_x^* M \otimes T_{\phi(x)}\tilde{M}$.

Definition 5.2. *If $M' \Subset M$ is a compact domain, where (M, g) has the canonical measure dx associated with g and $\phi \in C^\infty(M', \tilde{M})$, the* ***energy*** *of ϕ is defined as*

$$E(\phi, M') := \int_{M'} e(\phi)(x)\, dx, \quad dx := \sqrt{\det g}\, dx_1 \cdots dx_n \tag{5.3}$$

where $(\, x_\alpha\,)$ is a local coordinate on M. If M is compact, we write $E(\phi) := E(\phi, M)$.

Let $\phi^{-1}T\tilde{M}$ denote the bundle over M of which each fibre is the tangent space $T_{\phi(x)}\tilde{M}$ where $\phi \in C^\infty(M, \tilde{M})$, i.e.

$$\phi^{-1}T\tilde{M} := \bigcup_{x \in M} T_{\phi(x)}\tilde{M}.$$

Therefore, $\phi_* \in \Gamma(T^*M \otimes \phi^{-1}T\tilde{M})$. First of all, we need to give a connection in the bundle $T^*M \otimes \phi^{-1}T\tilde{M}$.

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on M and $\tilde{M}$ respectively. Now we can extend ∇ usually to T^*M as in the tensor bundle, i.e.

$$\nabla_X \omega(Y) := X\, \omega(Y) - \omega(\nabla_X Y)$$

for $\omega \in \Gamma(T^*M)$ and $X, Y \in \Gamma(TM)$. In terms of local coordinate $(\, x_\alpha\,)$ on M,

$$\nabla_{\frac{\partial}{\partial x_\alpha}} dx^\beta \left(\frac{\partial}{\partial x_\gamma}\right) = \frac{\partial}{\partial x_\alpha}\left[dx^\beta\left(\frac{\partial}{\partial x_\gamma}\right)\right] - dx^\beta\left(\nabla_{\frac{\partial}{\partial x_\alpha}} \frac{\partial}{\partial x_\gamma}\right) = -\Gamma^\beta_{\alpha\gamma},$$

which implies $\nabla_{\frac{\partial}{\partial x_\alpha}} dx^\beta = -\Gamma^\beta_{\alpha\gamma} dx^\gamma$.

Let (y_i) be a local coordinate on $\tilde{M}$. Define, without ambiguity, ∇ be the *induced connection* in $\phi^{-1}T\tilde{M}$ from $\tilde{\nabla}$ by

$$\nabla_{\frac{\partial}{\partial x_\alpha}}\left(\frac{\partial}{\partial y_i}\right) := \tilde{\nabla}_{\phi_*(\frac{\partial}{\partial x_\alpha})}\frac{\partial}{\partial y_i}$$

and extend it accordingly. Therefore,

$$\nabla_{\frac{\partial}{\partial x_\alpha}}\left(\frac{\partial}{\partial y_i}\right) = \tilde{\nabla}_{\frac{\partial \phi^j}{\partial x_\alpha}\frac{\partial}{\partial y_j}}\frac{\partial}{\partial y_i} = \frac{\partial \phi^j}{\partial x_\alpha}\tilde{\nabla}_{\frac{\partial}{\partial y_j}}\frac{\partial}{\partial y_i} = \frac{\partial \phi^j}{\partial x_\alpha}\tilde{\Gamma}^k_{ji}(\phi)\frac{\partial}{\partial y_k}$$

and

$$\left(\nabla_{\frac{\partial}{\partial x_\alpha}}\left(\frac{\partial}{\partial y_i}\right)\right)_x = \frac{\partial \phi^j}{\partial x_\alpha}(x)\tilde{\Gamma}^k_{ji}(\phi(x))\left(\frac{\partial}{\partial y_k}\right)_{\phi(x)}.$$

Moreover, the metric $\tilde{g}$ in $T\tilde{M}$ defines canonically a fibre metric in $\phi^{-1}T\tilde{M}$ and

$$\nabla_{\frac{\partial}{\partial x_\alpha}}\tilde{g} = \frac{\partial \phi^j}{\partial x_\alpha}\tilde{\nabla}_{\frac{\partial}{\partial y_k}}\tilde{g} = 0$$

since $\tilde{\nabla}$ is compatible with $\tilde{g}$. Thus ∇ is compatible with the fibre metric $\phi^*\tilde{g}$.

We now define ∇ be the connection in the bundle $T^*M \otimes \phi^{-1}T\tilde{M}$ as

$$\nabla_X(\omega \otimes \tilde{Y}) := (\nabla_X \omega) \otimes \tilde{Y} + \omega \otimes (\nabla_X \tilde{Y}) \tag{5.4}$$

for $X \in \Gamma(TM)$, $\omega \in \Gamma(T^*M)$ and $\tilde{Y} \in \Gamma(\phi^{-1}T\tilde{M})$. Then ∇ is a connection in $T^*M \otimes \phi^{-1}T\tilde{M}$; and it is compatible with the metric $g^{\alpha\beta}\tilde{g}_{ij}(\phi)$.

Given $\phi \in C^\infty(M, \tilde{M})$, the ***second fundamental form*** of ϕ is defined to be $\nabla\phi_* \in \Gamma(T^*M \otimes T^*M \otimes \phi^{-1}T\tilde{M})$ and $\nabla\phi_*(X,Y) := (\nabla_X\phi_*)(Y)$ for $X, Y \in \Gamma(TM)$. Moreover we have

$$\nabla_{\frac{\partial}{\partial x_\alpha}}\phi_* = \left(\frac{\partial^2\phi^i}{\partial x_\alpha \partial x_\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial \phi^i}{\partial x_\gamma} + \tilde{\Gamma}^i_{jk}(\phi)\frac{\partial\phi^k}{\partial x_\beta}\frac{\partial\phi^j}{\partial x_\alpha}\right) dx^\beta \otimes \frac{\partial}{\partial y_i}$$

which implies $\nabla\phi_*(X,Y) = \nabla\phi_*(Y,X)$ for $X, Y \in \Gamma(TM)$, or ${\phi^i}_{;\alpha,\beta} = {\phi^i}_{;\beta,\alpha}$ for $\alpha, \beta = 1, \dots, n$ and $i = 1, \dots, n$. Before defining the tension field, we review that given $u \in C^\infty(M)$, the *Laplacian* of u on M, denoted Δu, is defined by

$$\Delta u := \operatorname{tr}\operatorname{Hess}(u) = g^{\alpha\beta}\left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial u}{\partial x_\gamma}\right).$$

As an analogy, we define

Definition 5.3. *Given $\phi \in C^\infty(M, \tilde{M})$, the **tension field** of ϕ is defined as*

$$\tau(\phi) := \operatorname{tr}\nabla\phi_* \in \Gamma(\phi^{-1}T\tilde{M}) \tag{5.5}$$

where 'tr' *denotes the trace. If $\tilde{M} = \mathbb{R}$, then $\tau(\phi) = \Delta\phi\,\frac{d}{dy}$.*

In terms of local coordinates, $\tau(\phi) = \tau^i(\phi)\frac{\partial}{\partial y_i}$ is represented by

$$\begin{aligned}\tau^i(\phi) = g^{\alpha\beta}{\phi^i}_{;\alpha,\beta} &= g^{\alpha\beta}\left(\frac{\partial^2\phi^i}{\partial x_\alpha \partial x_\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial \phi^i}{\partial x_\gamma} + \tilde{\Gamma}^i_{jk}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial\phi^k}{\partial x_\beta}\right)\\ &= \Delta\phi^i + g^{\alpha\beta}\tilde{\Gamma}^i_{jk}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial\phi^k}{\partial x_\beta}.\end{aligned} \tag{5.6}$$

5.2 Harmonic Map and Euler-Lagrange Equation

Definition 5.4. *A smooth map $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ is* ***harmonic*** *if it is a critical point of the energy $E(\phi, M')$, for every compact subdomain $M' \Subset M$, defined in* (5.3).

Remark. If $M = (-\varepsilon, \varepsilon)$, then the harmonic maps on $(-\varepsilon, \varepsilon)$ are the geodesics in $\tilde{M}$ since it is a critical point of the length energy functional.

Let $\Phi : M \times (-\varepsilon, \varepsilon) \to \tilde{M}$ be a smooth variation of $\phi \in C^\infty(M, \tilde{M})$ defined by

$$\Phi(x, t) := \exp_{\phi(x)} tV(x)$$

where V is a vector field along ϕ with compact support inside $M' \Subset M$. By calculus of variation, one can derive the ***first variational formula*** of the energy.

Theorem 5.1. *With $\Phi(x, t) := \exp_{\phi(x)} tV(x)$ and a compact subdomain $M' \Subset M$ defined above,*

$$\frac{d}{dt} E(\Phi, M')\bigg|_{t=0} = \int_{M'} \langle \nabla V, \phi_* \rangle dx = -\int_{M'} \langle V, \tau(\phi) \rangle dx \tag{5.7}$$

where $dx = \sqrt{\det g}\, dx_1 \cdots dx_n$ is a canonical measure on M associated with g, and $\tau(\phi)$ is the tension field of ϕ defined in (5.5).

Definition 5.5. *A smooth map $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ is* ***harmonic*** *if*

$$\tau(\phi) = 0 \tag{5.8}$$

on M. The equation (5.8) *is called the* ***Euler-Lagrange equation****.*

Remark. A harmonic map ϕ defined in Definition 5.5 must be a harmonic map defined in Definition 5.4 due to the first variational formula for energy (5.7).

Now, given $\phi \in C^\infty(M, \tilde{M})$, $\psi \in C^\infty(\tilde{M}, \bar{M})$, $X, Y \in \Gamma(M)$, one can easily derive the following *composition formulae*,

$$(\nabla_X(\psi \circ \phi)_*)(Y) = (\tilde{\nabla}_{\phi_* X}\psi_*)(\phi_* Y) \; + \; \psi_*\big((\nabla_X \phi_*)(Y)\big)\,. \tag{5.9}$$

Taking the trace, one can get

$$\tau(\psi \circ \phi) = g^{\alpha\beta}(\tilde{\nabla}\psi_*)\left(\phi_*\frac{\partial}{\partial x_\alpha}, \phi_*\frac{\partial}{\partial x_\beta}\right) + \psi_*\big(\tau(\phi)\big) \tag{5.10}$$

If $\bar{M} = \mathbb{R}$, then (5.10) now reads

$$\Delta(\psi \circ \phi) = g^{\alpha\beta}\mathrm{Hess}(\psi)\left(\phi_*\frac{\partial}{\partial x_\alpha}, \phi_*\frac{\partial}{\partial x_\beta}\right) + \frac{\partial \psi}{\partial y_i}(\phi)\tau^i(\phi). \tag{5.11}$$

Lemma 5.1. *If $\psi \in C^\infty(\tilde{M}, \mathbb{R})$ is convex and $\phi \in C^\infty(M, \tilde{M})$ is harmonic, then $\psi \circ \phi$ is subharmonic, i.e. $\Delta(\psi \circ \phi) \geq 0$ on M.*

Proof. By virtue of (5.11) and since $\tau(\phi) = 0$, we have

$$\Delta(\psi \circ \phi) = g^{\alpha\beta}\mathrm{Hess}(\psi)\left(\phi_* \frac{\partial}{\partial x_\alpha}, \phi_* \frac{\partial}{\partial x_\beta}\right) \geq 0,$$

since ψ is convex. □

We thus have the following theorem from Jost [19, Corollary 8.7.6], which was useful in proving Theorem 4.2.

Corollary 5.1. *Let M be a compact manifold and $\phi : M \to \tilde{M}$ be a harmonic map. Assume that there exists a strictly convex function ψ on $\phi(M)$, then ϕ is constant.*

Definition 5.6. *A smooth map $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ is* ***totally geodesic*** *if its second fundamental form $\nabla \phi_* = 0$ on M.*

Lemma 5.2. *If $\psi \in C^\infty(\tilde{M}, \bar{M})$ is totally geodesic and $\phi \in C^\infty(M, \tilde{M})$ is harmonic, then $\psi \circ \phi$ is harmonic on M.*

Proof. By the composition formula (5.10). □

Corollary 5.2. *Let $\iota : (M, g) \hookrightarrow (\tilde{M}, \tilde{g})$ be an immersion. Then M is a totally geodesic submanifold in $\tilde{M}$ if and only if $\nabla \iota_* = 0$ on M.*

Proof. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic in M. If $M \hookrightarrow \tilde{M}$ is totally geodesic, then $\iota \circ \gamma$ is also a geodesic in $\tilde{M}$. By virtue of (5.10), we have

$$0 = \tau(\iota \circ \gamma) = (\nabla \iota_*)\Big(\gamma_* \frac{d}{dt}, \gamma_* \frac{d}{dt}\Big) + \iota_*\big(\tau(\gamma)\big) = (\nabla \iota_*)(\dot{\gamma}, \dot{\gamma})$$

where $\frac{d}{dt} := 1 \in \mathbb{R}$. Since $\nabla \iota_*$ is symmetric in $\Gamma(TM)$, $\nabla \iota_*$ vanishes on M. Conversely if $\nabla \iota_* = 0$ on M, from Lemma 5.2, $\iota \circ \gamma$ is also a geodesic in $\tilde{M}$. □

5.3 The Gauss Map and its Tension Field

Let (M, g) be an n-dimensional Riemannian manifold immersed in $\mathbb{R}^{n+m}$. The ***Gauss map*** γ on M is defined by sending each point $x \in M$ to its tangent n-plane viewed as an n-plane in $\mathbb{R}^{n+m}$, namely

$$\gamma : M \to G(n, m), \quad \gamma(x) := T_x M \subseteq \mathbb{R}^{n+m} \tag{5.12}$$

where $G(n, m)$ denotes the (oriented) Grassmannian. Now we have cf. [28],

Lemma 5.3. *$T_{\gamma(x)} G(n, m) \cong T_x^* M \otimes N_x M$ for all $x \in M$.*

Proof. We may identify $T_{\gamma(x)}G(n,m) \cong \mathbb{R}^{n\times m}$, which is spanned by $E_{\alpha i}$ defined in Chapter 4, where we adopt $\alpha = 1,\dots,n$ and $i = 1,\dots,m$.

Fix $e_1,\dots,e_n \in T_xM$ to be a basis for T_xM and $e_{n+1},\dots,e_{n+m}$ to be a basis for N_xM. We then identify $\mathbb{R}^{n+m} \cong T_x^*M \otimes N_xM$ by setting

$$E_{\alpha i} : e_\alpha \mapsto e_{n+i}. \tag{5.13}$$

Therefore $T_{\gamma(x)}G(n,m) \cong T_x^*M \otimes N_xM$. □

For our situation, we study in particular the graph M of f in $\mathbb{R}^{n+m}$.

Theorem 5.2. *Upon using the identification in Lemma* 5.3, *and we suppose* $M = \{\,(x, f(x)) : x \in \mathbb{R}^n\,\}$ *for some smooth map* $f : \mathbb{R}^n \to \mathbb{R}^m$ *then*

$$\gamma_* = \mathrm{II}$$

where II *denotes the second fundamental form of* M *in* $\mathbb{R}^{n+m}$.

Proof. Now $\gamma(x) = T_xM =: e_1 \wedge \cdots \wedge e_n$ where for $\alpha = 1,\dots,n$ and $i = 1,\dots,m$,

$$e_\alpha = \varepsilon_\alpha + \frac{\partial f^i}{\partial x_\alpha}(x)\varepsilon_{n+i}, \quad e_{n+i} = \varepsilon_{n+i} - \frac{\partial f^i}{\partial x_\alpha}(x)\varepsilon_\alpha$$

with e_{n+i} spans N_xM, and $\varepsilon_1,\dots,\varepsilon_{n+m}$ are the standard bases on $\mathbb{R}^{n+m}$ i.e.

$\gamma(x) = \left(\frac{\partial f^i}{\partial x_\alpha}(x) \right)$ in terms of local coordinate (4.7) on $G(n,m)$. Then

$$\gamma_* e_\alpha = \frac{\partial^2 f^i}{\partial x_\alpha \partial x_\beta}(x) E_{\beta i} \quad \Rightarrow \quad \gamma_*(e_\alpha, e_\beta) := (\gamma_* e_\alpha)(e_\beta) = \frac{\partial^2 f^i}{\partial x_\alpha \partial x_\beta}(x) e_{n+i}.$$

Let $\bar{\nabla}$ be the Euclidean connection on $\mathbb{R}^{n+m}$. Then

$$\bar{\nabla}_{e_\alpha} e_\beta = \bar{\nabla}_{\varepsilon_\alpha + \frac{\partial f^i}{\partial x_\alpha}(x)\varepsilon_{n+i}} \left(\varepsilon_\beta + \frac{\partial f^j}{\partial x_\beta}(x) \varepsilon_{n+j} \right) = \frac{\partial^2 f^j}{\partial x_\alpha \partial x_\beta}(x) \varepsilon_{n+j}.$$

However $\langle \varepsilon_{n+j}, e_{n+i} \rangle = \delta_{ij} \Rightarrow \varepsilon_{n+i}^\perp = e_{n+i}$. Hence $\gamma_*(e_\alpha, e_\beta) = \mathrm{II}(e_\alpha, e_\beta)$. □

The following theorem is one of the most famous results by Ruh and Vilms [28].

Theorem 5.3. *$\tau(\gamma) = \nabla H$, where $H := \mathrm{tr}_g \mathrm{II}$ is the* ***mean curvature vector field*** *on M, and ∇ is, without ambiguity, the normal connection on M in $\mathbb{R}^{n+m}$.*

Proof. Let ∇ be induced canonically from the Levi-Civita connection and normal connection, both denoted ∇, on M, we have for $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned}(\nabla_X \mathrm{II})(Y,Z) &:= \nabla_X(\mathrm{II}(Y,Z)) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z) \\ &= \nabla_X(\mathrm{II}(Z,Y)) - \mathrm{II}(Z, \nabla_X Y) - \mathrm{II}(\nabla_X Z, Y) = (\nabla_X \mathrm{II})(Z,Y).\end{aligned}$$

By Peterson-Codazzi equation,

$$\begin{aligned}(\nabla_X \mathrm{II})(Z,Y) - (\nabla_Z \mathrm{II})(X,Y) &= \left(\bar{R}(X,Z)Y \right)^\perp = 0 \\ (\nabla_X \mathrm{II})(Y,Z) &= (\nabla_Z \mathrm{II})(X,Y).\end{aligned}$$

Therefore,

$$\begin{aligned}\tau(\gamma) = \operatorname{tr} \nabla \gamma_* = g^{\alpha\beta}\left(\nabla_{\frac{\partial}{\partial x_\alpha}} \gamma_*\right)\left(\frac{\partial}{\partial x_\beta}\right) &= dx^\gamma \otimes g^{\alpha\beta}\left(\nabla_{\frac{\partial}{\partial x_\alpha}} \mathrm{II}\right)\left(\frac{\partial}{\partial x_\beta}, \frac{\partial}{\partial x_\gamma}\right) \\ = dx^\gamma \otimes g^{\alpha\beta}\left(\nabla_{\frac{\partial}{\partial x_\gamma}} \mathrm{II}\right)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) &= dx^\gamma \otimes \left(\nabla_{\frac{\partial}{\partial x_\gamma}} \operatorname{tr}_g \mathrm{II}\right) = \nabla H,\end{aligned}$$

where the connection ∇ commutes with the contraction tr_g. □

In conclusion, this famous theorem is renowned for its characterization of the harmonicity of the Gauss map, namely

Corollary 5.3. *Let M be an immersed submanifold in $\mathbb{R}^{n+m}$. The Gauss map on M is harmonic if and only if the mean curvature vector field of M is parallel.*

In particular,

Corollary 5.4. *The Gauss map of a minimal n-dimensional submanifold in $\mathbb{R}^{n+m}$ is harmonic.*

5.4 Simple Riemannian Manifolds and A Liouville-Type Result of Harmonic Maps

A Riemannian manifold (M, g) is said to be ***simple***, if there exist positive numbers $\lambda, \mu > 0$ such that

$$\lambda|\xi|^2 \leq g_{\alpha\beta}(x)\xi_\alpha\xi_\beta \leq \mu|\xi|^2 \tag{5.14}$$

for all $x, \xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ is the coordinate representation of points in M. In other words, M is homeomorphic to $\mathbb{R}^n$ furnished with a metric g of which the associated Laplace-Beltrami operator

$$\Delta := g^{\alpha\beta}\left(\frac{\partial^2}{\partial x_\alpha \partial x_\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial x_\gamma}\right)$$

is uniformly elliptic on $\mathbb{R}^n$.

By a classical Liouville theorem, a bounded entire harmonic function on $\mathbb{R}^n$ is necessarily a constant. Now we shall look at an analogous result [16, Theorem 1] due to Hildebrandt, Jost and Widman, for harmonic mappings of Riemannian manifolds.

Theorem 5.4. *Let ϕ be a harmonic map of a simple or compact Riemannian manifold M into a complete Riemannian manifold $\tilde{M}$, of which the sectional curvature is bounded from above by a constant $\kappa \geq 0$. Denote by $B_R(x_0)$ a geodesic* *ball in $\tilde{M}$ with radius $R < \frac{\pi}{2\sqrt{\kappa}}$ which does not meet the cut locus of its centre x_0. Assume also that the range $\phi(M)$ is contained inside $B_R(x_0)$. Then ϕ is a constant map.*

Indeed, Hildebrandt, Jost and Widman have also done an *a priori* estimate, cf. [16] Theorem 4 pp. 275 – 286, for harmonic maps in simple Riemannian manifold, so that Theorem 5.4 will then follow. Furthermore, as for the compact case, since the square of the distance function $d(\,\cdot, x_0)^2$ is a smooth convex function. By composition formula, $d(\,\phi(\,\cdot\,), x_0)^2$ is a subharmonic function and result follows from Corollary 5.1.

Remark. In Theorem 5.4, the geodesic ball $B_R(x_0)$ can be replaced by a geodesically convex neighbourhood around x_0 satisfying *the same condition*, that is, ϕ can also be shown similarly a constant map.

Chapter 6

Bernstein-Type Results for Special Lagrangian Graphs

This chapter focuses on the Bernstein-type results on special Lagrangian graphs in $\mathbb{C}^n$. We shall give a survey on the efforts made by Jost and Xin in their paper [21].

6.1 Convexity and Bounded Slope Assumption

Theorem 6.1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined on the whole $\mathbb{R}^n$. Assume that the graph of ∇F is a special Lagrangian submanifold M in $\mathbb{C}^n := \mathbb{R}^n \oplus \mathbb{R}^n$; cf.* (2.8). *Furthermore if* (*i*) *F is convex;*

(*ii*) *there is a constant $\beta < \infty$ such that*

$$\Delta_F^2 := \det(I + (\operatorname{Hess} F)^2) \leq \beta^2, \tag{6.1}$$

then F is a quadratic polynomial and M is an affine n-plane.

Proof of Theorem 6.1. Let $e_1, \ldots, e_{2n}$ be the standard bases for $\mathbb{R}^{2n}$. Choose P_0 as an n-plane $e_1 \wedge \cdots \wedge e_n$. At each point $x \in M$, the image n-plane by the Gauss map γ is $P := \gamma(x) = f_1 \wedge \cdots \wedge f_n \in \operatorname{Lag}(n)$ where

$$f_\alpha := e_\alpha + \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} e_{n+\beta}, \quad \alpha, \beta, \cdots = 1, \ldots, n.$$

Thus the induced metric g on M is

$$g_{\alpha\beta} := \langle f_\alpha, f_\beta \rangle = \delta_{\alpha\beta} + \frac{\partial^2 F}{\partial x_\alpha \partial x_\gamma} \frac{\partial^2 F}{\partial x_\beta \partial x_\gamma}.$$

Let $\mu_\alpha := \tan\theta_\alpha$ be the eigenvalues of $\operatorname{Hess} F$ which is symmetric as F is smooth. Clearly the eigenvalues of $(g_{\alpha\beta}) = 1 + \mu_\alpha^2 \geq 1$. Moreover,

$$\Delta_F^2 := \det(I + (\operatorname{Hess} F)^2) = \prod_\alpha (1 + \mu_\alpha^2) \leq \beta^2 \;\Rightarrow\; 1 + \mu_\alpha^2 \leq \beta^2. \tag{6.2}$$

This implies that M is a *simple* Riemannian manifold.

Since F is ***convex***, $\mu_\alpha \geq 0$. Thus (6.2) implies $0 \leq \mu_\alpha := \tan\theta_\alpha \leq \sqrt{\beta^2 - 1}$. By an action of $\mathrm{SO}(n) \times \mathrm{SO}(n)$, we may assume $P := (\delta_{\alpha i} \tan\theta_\alpha)$ in local coordinate (4.7), and from Wong's formula (4.11), we may set

$$\lambda_\alpha := \frac{\theta_\alpha}{\sqrt{\theta_1^2 + \cdots + \theta_n^2}} \geq 0, \quad \text{and} \quad t := \sqrt{\theta_1^2 + \cdots + \theta_n^2} \tag{6.3}$$

where $P := (\,\delta_{\alpha i} \tan \lambda_\alpha t\,)$. Now we have

$$\lambda_\alpha t = \theta_\alpha \leq \tan^{-1} \sqrt{\beta^2 - 1} \;\Rightarrow\; t \leq \frac{1}{\lambda_{\alpha'}} \tan^{-1} \sqrt{\beta^2 - 1}, \;\; \lambda_{\alpha'} := \max \lambda_\alpha.$$

Define in normal coordinate around P_0 inside $\mathrm{Lag}(n)$,

$$B_G(P_0) := \left\{ (X, t) : X = (\,\lambda_\alpha \delta_{\alpha i}\,), \lambda_\alpha \geq 0, 0 \leq t \leq t_X := \frac{1}{\lambda_{\alpha'}} \tan^{-1} \sqrt{\beta^2 - 1} \right\}$$

where $\sum_\alpha \lambda_\alpha^2 = 1$ and t denotes the arc-length parameter.

By construction $\gamma(M) \subseteq B_G(P_0)$, and from §4.4 it is geodesically convex.

If $\lambda_\alpha \geq 0$, by virtue of Table 4.1 the sectional curvature at P_0 is bounded above by

$$\max(\lambda_\alpha - \lambda_\beta)^2 \leq (\,\max \lambda_\alpha - \min \lambda_\beta\,)^2 \leq \lambda_{\alpha'}^2.$$

Therefore by (4.20), the square $d(\,\cdot\,, P_0)^2$ of the distance function from P_0 is a strictly convex function in $B_G(P_0)$. By (4.11), $B_G(P_0)$ does not meet the cut-locus of P_0, so similar reasoning of Theorem 5.4 implies γ is constant. $\square$

Remark. If the graph M of ∇F is a submanifold with parallel mean curvature instead of a minimal submanifold, Theorem 6.1 is still valid.

6.2 Spherical Bernstein-Type Result

A closed minimal hypersurface is a *hypersphere* if it is diffeomorphic to the sphere of codimension one. Calabi [5] and Almgren [2] have proved that the totally geodesic equator is the only minimal hypersurface in S^3. This is analogous to the classical Bernstein theorem and attracts Chern [7] to propose a *spherical Bernstein problem.* Many efforts are made to solve this analogous problem, for example one result by Jost and Xin [21] is the following:

Theorem 6.2. *Let M be a simple or compact minimal Legendrian submanifold in S^{2n-1}. Suppose that there is a fixed normal n-plane P_0 and some $\delta > 0$ such that*

$$\langle P, P_0 \rangle \geq \delta \tag{6.4}$$

holds for all normal n-planes P of M in S^{2n-1}. Then M is contained in a totally geodesic subsphere of S^{2n-1}.

Let $M \hookrightarrow S^{n+m} \hookrightarrow \mathbb{R}^{n+m+1}$ be an m-dimensional submanifold in the sphere. For each $x \in M$, by parallel translation in $\mathbb{R}^{n+m+1}$ the normal space N_xM of M in S^{n+m} is moved to $0 \in \mathbb{R}^{n+m+1}$ which can be viewed as an n-subspace in $\mathbb{R}^{n+m+1}$. Then we have defined the ***normal Gauss map*** $\gamma : M \to G(n, m+1)$.

In §3.2, to avoid singularity, we consider also the *associated truncated cone* CM_ε which is the image of $M \times [\varepsilon, \infty)$ on $\mathbb{R}^{n+m+1}$ under the map $(x, t) \mapsto tx$ for all $\varepsilon > 0$. Define a map $\psi : \mathbb{R}^{n+m+1} \setminus \{0\} \to S^{n+m}$ by $\psi(y) := \frac{y}{|y|}$. Then for any map $\phi : M \to \bar{M}$, we can associate to it a map $\tilde{\phi} : CM_\varepsilon \to \bar{M}$, called a *cone-like map*, defined by $\tilde{\phi} := \phi \circ \psi$. Conversely $\phi = \tilde{\phi} \circ \iota$, where $\iota : M \to CM_\varepsilon$ is the inclusion.

We choose an orthonormal frame field $\{ e_i, e_{m+\alpha} \}$ in S^{n+m} where $\{ e_{m+\alpha} \}$ spans $\Gamma(NM)$, we can define an orthonormal frame field $\{ E_i, E_{m+\alpha}, \frac{\partial}{\partial r} \}$ in $\mathbb{R}^{n+m+1}$ by parallel translating along rays from $0 \in \mathbb{R}^{n+m+1}$, in which $\{ E_i, \frac{\partial}{\partial r} \}$ spans $\Gamma(T(CM))$ i.e.

$$E_s = \frac{1}{r} e_s$$

where $r := \sum_{s=1}^{n+m+1} y_s^2$ is the distance function from origin. Therefore the tangent spaces along any rays from $0 \in \mathbb{R}^{n+m+1}$ are the same, i.e. the classical Gauss map $\tilde{\gamma} : CM_\varepsilon \to G(m+1, n)$ is a cone-like map.

Thus the normal Gauss map $\gamma : M \to G(n, m+1)$ maps $x \in M$ to a subspace in $\mathbb{R}^{n+m+1}$ spanned by $E_{m+\alpha}$. There is a natural isometry η between $G(n, m+1)$ and $G(m+1, n)$ defined by

$$\eta : P^n \mapsto (P^n)^\perp.$$

It is clear that $\eta \circ \gamma = \tilde{\gamma} \circ \iota$. In our situation, $m = n - 1$ and by Proposition 3.2 and §6.1, we may assume $\tilde{\gamma} : CM_\varepsilon \to \mathrm{Lag}(n)$ is a (Lagrangian) Gauss map and Theorem 5.4 can still apply due to Proposition 3.1 by Simons [29].

Moreover we have the following result by the discussion in §3.3.

Lemma 6.1. *The (Lagrangian) Gauss map $\tilde{\gamma} : CM_\varepsilon \to \mathrm{Lag}(n)$ has rank $n-1$ at most.*

Proof. Choose an orthonormal frame field $\{ E_u, \frac{\partial}{\partial r} \}$ in CM_ε as in §3.3.

By Proposition 5.2 and §3.3, we can compute that

$$\tilde{\gamma}_* E_u = \begin{pmatrix} {h_{st}}^u / r & 0 \\ 0 & 0 \end{pmatrix}_{n \times n} \quad \text{and} \quad \tilde{\gamma}_* \frac{\partial}{\partial r} = 0_{n \times n}$$

where ${h_{st}}^u$ are the coefficients of the second fundamental form of M in S^{2n-1} in the Je_u direction. Thus result follows. □

Proof of Theorem 6.2. We know that the Gauss map $\tilde{\gamma} : CM_\varepsilon \to \mathrm{Lag}(n)$ is harmonic due to Proposition 3.1. Moreover we have discussed that

$$\eta \circ \gamma = \tilde{\gamma} \circ \iota$$

where $\gamma : M \to G(n,n)$ is the normal Gauss map, $\iota : M \hookrightarrow CM_\varepsilon$ is the natural inclusion, and $\eta : G(n,n) \to G(n,n)$ is the natural isometry.

Let $\eta(P_0) := e_1 \wedge \cdots \wedge e_n$ which are complemented by n vectors e_{n+i}. By (6.4), $\langle \eta(P), \eta(P_0) \rangle = \langle P, P_0 \rangle \geq \delta$, for all normal n-planes P of M in S^{2n-1}.
Thus $\eta(P)$ lies in the coordinate neighbourhood around $\eta(P_0)$, and since $\eta(P) \in \mathrm{Lag}(n)$, $\eta(P)$ is symmetric by §4.5, we may assume $\eta(P) := (\, \mu_\alpha \delta_{\alpha i} \,) = (\, \delta_{\alpha i} \tan \theta_\alpha \,)$ in local coordinate (4.7) around $\eta(P_0)$. Clearly CM_ε is simple.

Now $\eta(P) \in \eta \circ \gamma(M) = \tilde{\gamma}(CM_\varepsilon)$, and let $Q(t) := f_1 \wedge \cdots \wedge f_n$ be the geodesic from $\eta(P_0)$ to $\eta(P)$ where

$$f_\alpha = e_\alpha + z_{\alpha i}\, e_{n+i}$$

where $Z(t) := (\, z_{\alpha i} \,) = (\, \delta_{\alpha i} \tan \lambda_\alpha t \,)$ is the local coordinate (4.7) of $Q(t)$ around $\eta(P_0)$, and $\dot{Z}(0) = (\, \lambda_\alpha \delta_{\alpha i} \,)$; cf. *Proof of Theorem 6.1.*

By right multiplication action of $\mathrm{SO}(n)$, we have at most one λ_α is negative. By Lemma 6.1, $\lambda_{\alpha''} = 0$ for some α''. Then by one more right multiplication of $\mathrm{SO}(n)$, $\lambda_\alpha \geq 0$ for all α.

Let $\tilde{f}_\alpha := (\cos \lambda_\alpha t)\, f_\alpha$, $\alpha := 1, \ldots, n$. Then $\tilde{f}_1, \ldots, \tilde{f}_n$ is orthonormal basis for

$T_{Q(t)}(CM_\varepsilon)$. Therefore,

$$\delta \leq \langle \eta(P_0), \eta(P) \rangle \leq \langle \eta(P_0), Q(t) \rangle = \prod_{\alpha=1}^{n} \cos \lambda_\alpha t \leq \cos \lambda_\alpha t$$

where $\lambda_\alpha \geq 0$ and $\sum_\alpha \lambda_\alpha^2 = 1$. Moreover, we may set $t \leq \frac{1}{\lambda_{\alpha'}} \cos^{-1} \delta$; cf. (4.11).

Define, in normal polar coordinates around $\eta(P_0)$,

$$\tilde{B}_G(P_0) := \left\{ (X, t) : X = (\lambda_\alpha \delta_{\alpha i}), \lambda_\alpha \geq 0, 0 \leq t \leq t_X := \frac{1}{\lambda_{\alpha'}} \cos^{-1} \delta \right\}.$$

By similar reasons in *Proof of Theorem 6.1*, we know $\tilde{\gamma}$ is constant, so is γ. □

Remark. The cone structure here can be used to compensate the condition that F is convex in Theorem 6.1.

6.3 Bernstein-Type Result with only Bounded Slope

We end this chapter by presenting the last theorem by Jost and Xin [21], where the condition of convexity of F is dropped.

Theorem 6.3. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined on the whole $\mathbb{R}^n$. Assume that the graph of ∇F is a special Lagrangian submanifold M in*

$\mathbb{C}^n := \mathbb{R}^n \oplus \mathbb{R}^n$; *cf.* (2.8). *Furthermore if there is a constant $\beta < \infty$ such that*

$$\Delta_F^2 := \det(I + (\operatorname{Hess} F)^2) \leq \beta^2, \tag{6.1}$$

then F is a quadratic polynomial and M is an affine n-plane.

Proof. We can consider the *tangent cone* of M at ∞ as Fleming in [12]. Take the intersection of M with ball of radius t and contract by $\frac{1}{t}$ to get a family of minimal submanifolds in the unit ball with submanifolds of S^{2n-1} as boundaries. More precisely, we define a sequence

$$F^t(x) := \frac{1}{t^2}\, F(tx).$$

For each t,

$$\frac{\partial^2 F^t}{\partial x_\alpha \partial x_\beta} = \frac{\partial^2 F}{\partial u_\alpha \partial u_\beta}$$

where $u_\alpha := t x_\alpha$. Hence F^t also satisfies (2.8) and (6.1). Moreover there exists a subsequence $t_j \to \infty$ such that

$$\lim_{t_j \to \infty} F^t(x) = \tilde{F}(x).$$

Then $\tilde{F}$ also satisfies (2.8) and (6.1) and the graph of $\nabla \tilde{F}$ is a special Lagrangian cone $C\tilde{M}$ in $\mathbb{R}^{2n}$ of which the link $\tilde{M}$ is a compact minimal Legendrian submanifold in S^{2n-1}.

Let $\{ e_\alpha, e_{n+\beta} \}$ be the standard bases of $\mathbb{R}^{2n}$. Choose $Q_0 := e_1 \wedge \cdots \wedge e_n \in \mathrm{Lag}(n)$. At each point x of $C\tilde{M}$, its image n-plane Q under the Gauss map is spanned by

$$f_\alpha = e_\alpha + \frac{\partial^2 \tilde{F}}{\partial x_\alpha \partial x_\beta} e_{n+\beta}$$

where

$$| f_1 \wedge \cdots \wedge f_n |^2 = \det(I + (\mathrm{Hess}\, F)^2) = \Delta_F^2$$

and the n-plane Q is also spanned by

$$\tilde{f}_\alpha = \Delta_F^{-1/n} f_\alpha, \qquad | \tilde{f}_1 \wedge \cdots \wedge \tilde{f}_n | = 1.$$

Thus we have

$$\langle Q, Q_0 \rangle = \det (\langle e_\alpha, \tilde{f}_\beta \rangle) = \Delta_F^{-1} \geq \beta^{-1}.$$

Let η be the natural isometry in $G(n,n)$ defined above. Thus we have

$$\langle \eta(Q), \eta(Q_0) \rangle \geq \beta^{-1},$$

where $\eta(Q)$ is just the normal n-plane of $\tilde{M}$ in S^{2n-1}. By Theorem 6.2, $\tilde{M}$ is a totally geodesic subsphere S^{n-1} in S^{2n-1}, therefore $C\tilde{M}$ is an n-plane in $\mathbb{R}^{2n}$. By Allard's regularity estimate [1], M is an affine n-plane and F is a quadratic polynomial. $\square$

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